AN ELEMENTARY PROOF OF THE STRONG FORM OF THE CAUCHY THEOREM

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A difficult step in the derivation of the strong forms of the Cauchy theorem, Green's lemma, and related theorems from the corresponding weak forms is the construction, for a given rectifiable Jordan curve J, of a sequence of Jordan polygons lying interior to J, converging to J, and having uniformly bounded lengths. This note presents what the author believes to be a simpler elementary construction of this sequence than any hitherto available (see [1, 2][1] and the bibliographies at the ends of these papers). We shall illustrate its use by proving the strong Cauchy theorem.

We assume the Jordan separation theorem together with its elementary consequences, as is usual in proofs of this kind, and we assume the weak form of the Cauchy theorem ($\int_P f(z)dz=0$ if $f(z)$ is analytic in a region containing P and its interior) for Jordan polygons P whose edges lie on lines of the form $x=m2^{-N}$, $y=n2^{-N}$, where $m, n=0, \pm 1, \cdots$. The polygon P and its interior is of course the sum of a finite number of squares of the network of closed squares (with sides 2^{-N}) into which these lines cut up the plane, so that we need to assume the weak Cauchy theorem only for single squares.

Let X be a fixed interior point of the given rectifiable Jordan curve J with coordinates *not* of the form $m2^{-N}$, and let C be a fixed closed square lying interior to J, containing X and having sides parallel to the axes. Let I be the set of squares of the above network lying interior to J; for sufficiently large N every square of I which contains a point of C lies interior to J. The vertical ray from X cuts a first edge l_1 of a square not in I. We form a polygon P by proceeding to the left along l_1 from its right-hand end point p_0, and at any vertex p_n choosing as l_{n+1} from the three remaining edges the counter-clockwise-most one which has a square of I on its left and a square not in I (and so containing a point of J) on its right. We shall call the latter square S_{n+1}. The reader will find the following argument trivial if he will sketch the four possible configurations at p_n. In tracing P a first vertex p_n must be repeated, $p_n=p_{n+M}$. If $n>0$, then l_n and l_{n+M} approach p_n from opposite directions, $S_{n+1}=S_{n+M}$, and the Jordan polygon l_{n+1} to l_{n+M} separates S_{n+1} from S_n and so separates points of J, an impossibility. Thus $n=0$, $p_M=p_0$ and P is a Jordan polygon. Note

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

that P depends on N, although it is not explicit in the notation.

Let q_n be a point of J in S_n nearest to l_n, and let q_nt_n be the perpendicular dropped from q_n to l_n (t_n on l_n). Because of the definition of the q's no two distinct segments q_nt_n and q_mt_m can intersect except possibly in a common end point; in particular, the order of the t_n on P is the same as that of the q_n on J.

As a general remark, the squares touching any S_n have a total of 24 edges, so that, for some $k \leqq 24$, S_{n+k} does not touch S_n and the length of the arc q_nq_{n+k} of J is at least 2^{-N}. Thus if indices n_i are chosen so that $24 \leqq n_{i+1} - n_i < 48$ for $i = 0, \cdots, K-1$ and $t_{n_0} = t_{n_K} = t_0$, then the length of the arc $q_{n_i}q_{n_{i+1}}$ of J is at least 2^{-N} and the length of the arc $t_{n_i}t_{n_{i+1}}$ of P is at most $48/2^N$. If C_i is the Jordan curve consisting of these two arcs and the two segments $q_{n_i}t_{n_i}$, $q_{n_{i+1}}t_{n_{i+1}}$, we have that $L(C_i) \leqq 51 \cdot L(q_{n_i}q_{n_{i+1}})$, and

$$\sum_{i=0}^{K-1} L(C_i) \leqq 51 \cdot L(J).$$

We can now apply the device of Newman [3, p. 156] to prove the strong Cauchy theorem.

THEOREM. *If $f(z)$ is analytic interior to J and continuous on J plus its interior, then $\int_J f(z)dz = 0$.*

The distance from q_{n_i} to $q_{n_{i+1}}$ is for every i at most $50/2^N$, and since J is rectifiable it follows that given δ there is an N such that for every i one of the two arcs determined on J by q_{n_i} and $q_{n_{i+1}}$ has length at most δ. But this must be the arc we have called $q_{n_i}q_{n_{i+1}}$ (containing q_j for $n_i \leqq j \leqq n_{i+1}$); for the length of its complementary arc on J is at least 1/48th of the length of the major arc of the points t_{n_i} and $t_{n_{i+1}}$ on P, and since the square C lies interior to P this length is bounded below by one-half the length of the perimeter of C. Thus the maximum diameter of the curves C_i approaches 0 as $N \to \infty$ and since $f(z)$ is uniformly continuous on J plus its interior, there exists for every ϵ an N such that $|f(z) - f(t_{n_i})| < \epsilon$ for all z on C_i. Now $\int_J f(z)dz = \sum_0^{K-1} \int_{C_i} f(z)dz$, for the difference between these two integrals consists of $\int_P$, which is 0 by hypothesis, and the integrals over the segments $p_{n_i}t_{n_i}$, which are traced once in each direction and so cancel. Since $\int_{C_i} f(t_{n_i})dz = f(t_{n_i}) \int_{C_i} dz = 0$, this implies that

$$\left| \int_J f(z)dz \right| = \left| \sum_{i=0}^{K-1} \int_{C_i} (f(z) - f(t_{n_i}))dz \right| \leqq \epsilon \cdot 51 L(J)$$

for every ϵ; thus $\int_J f(z)dz = 0$.

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ON A CERTAIN TYPE OF NONLINEAR INTEGRAL EQUATIONS

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1. **Introduction.** The object of this paper is to prove that the nonlinear integral equation

$$\phi(x) = \lambda\left[f(x) + \sum_{i=1}^{n}\int_a^b \cdots \int_a^b K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \phi(s_1), \cdots, \phi(s_i))ds_1 \cdots ds_i\right] \tag{1}$$

has at least one eigenvalue, provided the functionals

$$G_i(x, v) = \int_a^b \cdots \int_a^b K_i(x, s_1, \cdots, s_i) \cdot \bar{F}_i(s_1, \cdots, s_i, v(s_1), \cdots, v(s_i))ds_1 \cdots ds_i \tag{2}$$

are fully continuous, and the F_i satisfy a certain linear integrodifferential equation. The solution of (1) is shown to be equivalent to that of a variational problem containing infinitely many parameters. The latter problem, however, can be solved easily by the method of Rayleigh-Ritz, which consists in approaching the solution of the variational problem by a sequence of variational problems containing only a finite number of parameters. The convergence of this procedure is assured by a convergence theorem of Friedrich Riesz.

2. **Preparatory remarks.** Let I be the closed interval $a \leqq x \leqq b$, and L^2 the class of all functions having Lebesgue integrable squares on I with a norm not larger than N^2. Let, further, $\{v_n(x)\}$ $(n = 1, 2, 3, \cdots)$

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be a set of functions in L^2 and $\bar{v}(x)$ a function such that[1]

$$\lim_{n\to\infty} \int v_n(x)w(x)dx = \int \bar{v}(x)w(x)dx \tag{3}$$

for any arbitrary w of integrable square on I or, what is equivalent, for any arbitrary $w \in L^2$; that is, we assume the weak convergence of $\{v_n\}$, in the following denoted by $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$. We next show that $\bar{v} \in L^2$. Since the right-hand side of (3) is assumed to exist for every $w \in L^2$, it follows by a known result (see, for example, Kaczmarz-Steinhaus, *Theorie der Orthogonalreihen*) that $\bar{v}$ is of integrable square over I. Hence, if $w = \bar{v}$, (3) becomes

$$\int \bar{v}^2(x)dx = \lim_{n\to\infty} \int v_n(x)\bar{v}(x)dx,$$

and since by Schwarz's inequality

$$\left[\int v_n(x)\bar{v}(x)dx\right]^2 \leqq N^2\left[\int \bar{v}^2(x)dx\right],$$

we obtain, as claimed,

$$\int \bar{v}^2(x)dx \leqq N^2. \tag{4}$$

We now assume that the functional $G_i(x, v)$ be fully continuous, that is, that

$$\lim_{n\to\infty} G_i(x_n, v_n) = G_i(\bar{x}, \bar{v}), \qquad i = 1, 2, \cdots, m, \tag{5}$$

for any $\{x_n\} \in I$ and $v_n \in L^2$ for which $\lim_{n\to\infty} x_n = \bar{x}$ and $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$.

The introduction of a closed orthonormal system of functions $\{w_\nu(x)\} \in L^2$ associates with each v_n an infinite sequence of numbers

$$c_{n\nu} = \int v_n(x)w_\nu(x)dx, \qquad \nu = 1, 2, \cdots,$$

such that

$$\sum_{\nu=1}^{\infty} c_{n\nu}^2 = \int v_n^2(x)dx \leqq N^2$$

for every $n \geqq 1$. The class L^2 of functions v then corresponds to a class $\mathfrak{H}^2$ of vectors $\mathfrak{v} \equiv (c_1, c_2, \cdots)$ with $c_\nu = \int v(x)w_\nu(x)dx$. The transition

[1] All integrations are to be extended over the interval I.

from L^2 to $\mathfrak{H}^2$ implies the substitution in $G_i(x_n, v_n)$ of $v_n(x)$ by its equivalent $v(x) \sim \sum_{\nu=1}^{\infty} c_\nu w_\nu(x)$, and (5) now changes to

$$\lim_{n\to\infty} G_i(x_n, v_n) = G_i(\xi, v) \equiv P_i(\xi, c_1, c_2, \cdots)$$

for any $\{x_n\} \in I$ and $\{v_n\} \in \mathfrak{H}^2$ for which $\lim_{n\to\infty} x_n = \xi$ and $\lim_{n\to\infty} c_{n\nu} = c_\nu$, $\nu = 1, 2, \cdots$. From the full continuity of the G_i thus follows the full continuity (*Vollstetigkeit*)[2] of the P_i. It is just as easily seen that the converse also holds true.

3. **Construction of a solution of the integral equation.** In the G_i we now admit as arguments v_n only aggregates of the form $v_n(x) = \sum_{\nu=1}^{n} c_{n\nu} w_\nu(x)$ with $\sum_{\nu=1}^{n} c_{n\nu}^2 = N^2$ for every fixed n. The functional

$$(6) \qquad J(v_n) = 2\left[\int f(x)v_n(x)dx + \sum_{i=1}^{m} e_i \int G_i(x, v_n)v_n(x)dx\right]$$

—here the e_i denote finite numbers to be determined later—is now a continuous function of the $c_{n\nu}$ and so has at least one minimum. Let $c_{n\nu} = a_{n\nu}$ $(\nu = 1, 2, \cdots, n)$ be the minimal coordinates:

$$(7) \qquad \min J(v_n) = d_n = J(\phi_n), \quad \phi_n(x) = \sum_{\nu=1}^{n} a_{n\nu} w_\nu(x), \qquad \sum_{\nu=1}^{n} a_{n\nu}^2 = N^2.$$

As a consequence of (7) we have[3]

$$(8) \qquad \frac{\partial}{\partial c_{n\nu}}\left[J(v_n) + \frac{1}{\lambda_n}\left(N^2 - \sum_{\mu=1}^{n} c_{n\mu}^2\right)\right]_{c_{n\mu}=a_{n\mu}} = 0, \qquad \nu = 1, 2, \cdots, n.$$

However,

$$\frac{\partial J(v_n)}{\partial c_{n\nu}} = 2\left[\int f w_\nu dx + \sum_i e_i \int\int \cdots \int K_i\left(v_n \frac{\partial F_i}{\partial c_{n\nu}} + w_\nu F_i\right) dx ds_1 \cdots ds_i\right].$$

We must now make the following assumption: *The F_i satisfy the linear integrodifferential equations*

$$(9) \qquad \int\int \cdots \int K_i\left[e_i v_n \frac{\partial F_i}{\partial c_{n\nu}} - (1 - e_i) w_\nu F_i\right] dx ds_1 \cdots ds_i = 0$$

[2] See D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, 1912, p. 177. A fully continuous function $P(x, c_1, c_2, \cdots)$, where $x \in I$ and $\sum_{\nu=1}^{\infty} c_\nu \leq N^2$, is bounded.

[3] In (8), $1/\lambda_n$ denotes Lagrange's multiplier for the extremum problem under consideration. As will be shown subsequently, $\lambda_n \neq 0$ for every n.

for all arguments $v_n = \sum_{(\nu)} c_{n\nu} w_\nu$, *identically in the* $c_{n\nu}$.

In this case we obtain

$$\frac{\partial J(v_n)}{\partial c_{n\nu}} = 2\left[\int \left(f + \sum_i \int \cdots \int K_i F_i ds_1 \cdots ds_i\right) w_\nu dx\right],$$

and (8) leads to

$$a_{n\nu} = \lambda_n \int G(x, \phi_n) w_\nu dx, \qquad \nu = 1, 2, \cdots, n, \tag{10}$$

with

$$G(x, \phi_n) = f(x) + \sum_i \int \cdots \int K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \phi_n(s_1), \cdots, \phi_n(s_i)) ds_1 \cdots ds_i.$$

On account of $a_{n\nu} = \int \phi_n w_\nu dx$ the relations (10) may be written as

$$\int (\phi_n - \lambda_n G(x, \phi_n)) w_\nu dx = 0 \qquad \text{for } \nu = 1, 2, \cdots, n. \tag{11}$$

Equations (10) show that the $|\lambda_n|$ have a common positive lower bound: multiplication of (10) by $a_{n\nu}$ and summation for $\nu = 1, \cdots, n$ result in

$$N^2 = \lambda_n \int G(x, \phi_n) \phi_n dx. \tag{12}$$

But since $G(x, \phi_n)$ is a fully continuous function of the $a_{n\nu}$ for $x \in I$ and $\phi_n \in H^2$ it is bounded: there exists a $\delta > 0$ such that

$$|G(x, \phi_n)| \leqq N/(b-a)^{1/2}\delta \quad \text{for every } n.$$

Therefore

$$\left|\int G(x, \phi_n)\phi_n dx\right| \leqq \left[\int G^2(x, \phi_n) dx\right]^{1/2} \left[\int \phi_n^2 dx\right]^{1/2} \leqq N^2/\delta,$$

whence $|\lambda_n| \geqq \delta > 0$ for every n.

Now the $\int \phi_n^2 dx$ all have the same value N^2. This property of the sequence $\{\phi_n\}$ guarantees the existence of a $\bar{\phi}(x)$—defined almost everywhere in I and possessing a Lebesgue integrable square—which is the W-lim of a suitably chosen subsequence $\{\phi_{\bar{n}}\}$ of $\{\phi_n\}$,[4]

[4] Friedrich Riesz, *Untersuchungen über Systeme integrierbarer Funktionen*, Math. Ann. vol. 69 (1910) p. 467. The sequence $\{\bar{n}\}$ is determined by Hilbert's diagonal method (see, for example, Hellinger-Toeplitz, Encyklopädie der mathematischen Wissenschaften vol. II C 13, p. 1405).

$$W\text{-}\lim_{\bar{n}\to\infty} \phi_{\bar{n}}(x) = \bar{\phi}(x); \tag{13}$$

because the system $\{w_\nu\}$ is closed $\bar{\phi}$ is determined uniquely almost everywhere in I. On account of (4), $\int\bar{\phi}^2(x)dx \leqq N^2$.

We are now going to show that

$$\lim_{\bar{n}\to\infty} \int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx = \int G(x, \bar{\phi})\bar{\phi}dx. \tag{14}$$

Since

$$\left|\int G(x, \bar{\phi})\bar{\phi}dx - \int G(x, \phi_{\bar{n}})\phi_{\bar{n}}dx\right| \leqq \left|\int G(x, \bar{\phi})(\bar{\phi} - \phi_{\bar{n}})dx\right| + \left|\int (G(x, \bar{\phi}) - G(x, \phi_{\bar{n}}))\phi_{\bar{n}}dx\right|,$$

and the first expression on the right hand side—by (13)—may be made as small as desired by taking $\bar{n}$ sufficiently large, only the second term remains to be considered. Now

$$\left|\int (G(x, \bar{\phi}) - G(x, \phi_{\bar{n}}))\phi_{\bar{n}}dx\right| \leqq N\left[\int (G(x, \bar{\phi}) - G(x, \phi_{\bar{n}}))^2dx\right]^{1/2},$$

and so (14) will be proved if we can show that $\lim_{n\to\infty}\int(G(x, \bar{\phi}) - G(x, \phi_n))^2dx = 0$. This, however, follows immediately from the convergence theorem of Lebesgue.[5] The sequence $L_n \equiv (G(x,\bar{\phi}) - G(x, \phi_n))^2$ obviously satisfies the conditions of that theorem: (a) L_n is Lebesgue integrable; (b) Since $|G(x, v)| \leqq N/(b-a)^{1/2}\delta$, $|L_n| \leqq 4N^2/(b-a)\delta^2$ for every $\bar{n}$; (c) Because of the full continuity of $G(x, v)$, $\lim_{n\to\infty}(G(x, \bar{\phi}) - G(x, \phi_n)) = 0$. Therefore $\bar{L} = 0$, which proves (14).

We must now distinguish between these two cases:

I. There exists a $\delta' > 0$ such that $\left|\int G(x, \bar{\phi})\bar{\phi}dx\right| \geqq N^2/\delta'$;

II. $\int G(x, \bar{\phi})\bar{\phi}dx = 0$.

CASE I. By (12), $\lambda_n = N^2/\int G(x, \phi_n)\phi_n dx$, so that by (14)

$$\lim_{\bar{n}\to\infty} \lambda_{\bar{n}} = \bar{\lambda} = \frac{N^2}{\int G(x, \bar{\phi})\bar{\phi}dx} \tag{15}$$

exists; because of I, $|\bar{\lambda}| \leqq \delta'$.

[5] If a sequence of Lebesgue integrable functions $L_n(x)$ possessing a common bound has a limit function $\bar{L}(x)$, then $\bar{L}$, too, is Lebesgue integrable and $\lim_{n\to\infty}\int L_n(x)dx = \int\bar{L}(x)dx$.

If we now apply equations (11) to indices $\bar{n}$ only and then take the $\lim_{\bar{n}\to\infty}$ we obtain

$$\int (\bar{\phi}(x) - \bar{\lambda}G(x, \bar{\phi}))w_\nu(x)dx = 0 \qquad \text{for } \nu = 1, 2, \cdots .$$

Since the system of the $\{w_\nu\}$ is closed we may deduce $\bar{\phi} - \bar{\lambda}G(x, \bar{\phi}) = 0$, that is,

$$\bar{\phi}(x) = \bar{\lambda}\left[f(x) + \sum_{i=1}^{m}\int \cdots \int K_i(x, s_1, \cdots, s_i) \cdot F_i(s_1, \cdots, s_i, \bar{\phi}(s_1), \cdots, \bar{\phi}(s_i))ds_1 \cdots ds_i\right]$$

almost everywhere in I. We have thus obtained a solution $\bar{\phi}(x)$ of (1) belonging to the finite eigenvalue $\bar{\lambda}$.

The previously derived relationship $\int\bar{\phi}^2(x)dx \leqq N^2$ may now be improved: replacing $G(x, \bar{\phi})$ in (15) by its equal $(1/\bar{\lambda})\bar{\phi}$ leads to $\int\bar{\phi}^2(x)dx = N^2$.

CASE II. We write equations (10) for indices $\bar{n}$ only:

$$\int G(x, \phi_{\bar{n}})w_\nu dx = \frac{1}{\lambda_{\bar{n}}}a_{\bar{n}\nu}, \qquad \nu = 1, 2, \cdots, \bar{n},$$

and increasing $\bar{n}$ beyond any bound we obtain, since $\lim_{\bar{n}\to\infty}\lambda_{\bar{n}} = \infty$ and $|a_{\bar{n}\nu}| \leqq N$,

$$\int G(x, \bar{\phi})w_\nu dx = 0, \qquad \nu = 1, 2, \cdots .$$

In this case $\bar{\phi}(x)$ may be considered a solution of (1) belonging to $\lambda = \infty$.

4. **The variational problem.** We see, then, that $\bar{\phi}$ is always a solution of (1). This function possesses another important property: If $\bar{\mathfrak{H}}^2$ denotes the class of all $\mathfrak{v}(x) \sim \sum_{(\nu)} c_\nu w_\nu(x)$ with $\sum_{(\nu)} c_\nu^2 = N^2$, and

$$\mathfrak{v}(x) \sim \sum_{\nu=1}^{\infty} \bar{a}_\nu w_\nu(x) \quad \text{with} \quad \bar{a}_\nu = \int \bar{\phi}(x)w_\nu(x)dx,$$

then $\bar{\mathfrak{v}}$ minimizes $J(\mathfrak{v})$.

To prove this we notice first that $J(\mathfrak{v}_n)$ results from $J(\mathfrak{v}_{n+1})$ if we put $c_{n+1} = 0$. Let d_n be the minimum of $J(\mathfrak{v}_n)$ in $\bar{\mathfrak{H}}^2$. Then obviously $d_n \geqq d_{n+1}$. Let, further, d be the minimum of $J(\mathfrak{v})$ for $\mathfrak{v} \in \bar{\mathfrak{H}}^2$; then $d_n \geqq d$ for every n. Therefore, if $\bar{d} = \lim_{\bar{n}\to\infty} d_{\bar{n}} = J(\bar{\mathfrak{v}})$, we get $\bar{d} \geqq d$ or $\bar{d} = d + \eta$ with $\eta \geqq 0$. We shall show that $\eta = 0$.

Since d is the lower bound of $J(\mathfrak{v})$ in $\overline{\mathfrak{H}}^2$ there exists a $\mathfrak{p}(x) \sim \sum_{(\nu)} p_\nu w_\nu(x)$ in $\overline{\mathfrak{H}}^2$ so that $J(\mathfrak{p}) = \bar{d} - \theta\eta$ with $0 < \theta \leqq 1$. If, now, $\epsilon > 0$ be chosen as small as desired, there is, because of the full continuity of $J(\mathfrak{v})$, a $\delta > 0$ and an index r such that $|J(\mathfrak{v}) - J(\mathfrak{p})| < \epsilon$ for every $\mathfrak{v} \in \overline{\mathfrak{H}}^2$, so long as $|c_\nu - p_\nu| < \delta$ for $\nu = 1, 2, \cdots, r$. We take $\epsilon = \theta\eta$ and choose $r' \geqq r$ large enough to have $\sum_{\nu=1}^{r'} p_\nu^2 = N'^2 > N^2 \cdot N^2/(N+\delta)^2$. Then the vector $\bar{\mathfrak{p}}(x) = \sum_{\nu=1}^{r'} \bar{p}_\nu w_\nu(x)$ with $\bar{p}_\nu = (N/N')p_\nu$ belongs to $\overline{\mathfrak{H}}^2$, and since $N' > N^2/(N+\delta)$,

$$|\bar{p}_\nu - p_\nu| = |p_\nu| \cdot (N/N' - 1) \leqq N(N/N' - 1) < \delta$$

for $\nu = 1, 2, \cdots, r'$. We may, therefore, conclude that $|J(\bar{\mathfrak{p}}) - J(\mathfrak{p})| < \theta\eta$ or $J(\bar{\mathfrak{p}}) < J(\mathfrak{p}) + \theta\eta = \bar{d}$. But $d_{r'} \leqq J(\bar{\mathfrak{p}})$, and so $d_{r'} < \bar{d}$.

By now choosing $\bar{n}$, $\bar{n} \geqq r'$, such that $d_{\bar{n}} \leqq d_{r'}$ we get $d_{\bar{n}} \leqq \bar{d}$, a relation which contradicts the fact that the sequence d_n converges to $\bar{d}$ from above. Thus we see that $\eta = 0$ or $\bar{d} = J(\mathfrak{d})$.

5. Solution of the integrodifferential equation. It is easy to verify that equations (9) are fulfilled if we put $e_i = 1/(i+1)$, K_i continuous and

$$K_i(x, s_1, \cdots, s_k, \cdots, s_i) = K_i(s_k, s_1, \cdots, x, \cdots, s_i), \qquad k = 1, 2, \cdots, i,$$

$$F_i(s_1, \cdots, s_i, u_1, \cdots, u_i) = a_i u_1 \cdots u_i$$

for $i = 1, 2, \cdots, m$.

It remains to be shown that functionals of the type

$$Q(x, v) = \int \cdots \int K(x, s_1, \cdots, s_i) v(s_1) \cdots v(s_i) ds_1 \cdots ds_i$$

are fully continuous for $x \in I$ and $v \in L^2$. Let us, therefore, assume that $\{x_n\} \in I$, $\{v_n\} \in L^2$, $\lim_{n\to\infty} x_n = \bar{x}$, and $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$. Then

$$\begin{aligned} |Q(\bar{x}, \bar{v}) - Q(x_n, v_n)| &\leqq |Q(\bar{x}, \bar{v}) - Q(\bar{x}, v_n)| + |Q(\bar{x}, v_n) - Q(x_n, v_n)| \\ &= \left| \int \cdots \int K(\bar{x}, s_1, \cdots, s_i) [\bar{v}(s_1) \cdots \bar{v}(s_i) \right. \\ &\qquad \left. - v_n(s_1) \cdots v_n(s_i)] ds_1 \cdots ds_i \right| \\ &\quad + \left| \int \cdots \int [K(\bar{x}, s_1, \cdots, s_i) \right. \\ &\qquad \left. - K(x_n, s_1, \cdots, s_i)] v_n(s_1) \cdots v_n(s_i) ds_1 \cdots ds_i \right|. \end{aligned}$$

Because of the continuity of K the second term on the right-hand side may be made arbitrarily small by choosing n sufficiently large. In order to show that the same applies also to the first term we continue as follows:

$$\left| \int \cdots \int K[\bar{v}(s_1) \cdots \bar{v}(s_i) - v_n(s_1) \cdots v_n(s_i)]ds_1 \cdots ds_i \right|$$

$$= \left| \int \cdots \int K \sum_{k=1}^{i} \bar{v}(s_1) \cdots \bar{v}(s_{k-1})[\bar{v}(s_k) - v_n(s_k)] \right.$$

$$\left. \cdot v_n(s_{k+1}) \cdots v_n(s_i)ds_1 \cdots ds_i \right|$$

$$\leqq \sum_{k=1}^{i} \left| \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right) \right.$$

$$\left. \cdot \bar{v}(s_1) \cdots \bar{v}(s_{k-1})v_n(s_{k+1}) \cdots v_n(s_i)ds_1 \cdots ds_i \right|$$

$$\leqq \sum_{k=1}^{i} \left\{ \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right)^2 \right.$$

$$\left. \cdot ds_1 \cdots ds_{k-1}ds_{k+1} \cdots ds_i \right\}^{1/2} \cdot N^{i-1}.$$

Since $W\text{-}\lim_{n\to\infty} v_n = \bar{v}$, $\lim_{n\to\infty} (\int K[\bar{v}(s_k) - v_n(s_k)]ds_k) = 0$, and since $|\int K(x, s_1, \cdots, s_i)[\bar{v}(s_k) - v_n(s_k)]ds_k| \leqq 2N(b-a) \cdot \max|K|$, we see that the sequence of Lebesgue integrable functions $(\int K(x, s_1, \cdots, s_i) \cdot [\bar{v}(s_k) - v_n(s_k)]ds_k)^2$ has a common bound and the limit function zero. By Lebesgue's convergence theorem we may conclude that

$$\lim_{n\to\infty} \int \cdots \int \left(\int K[\bar{v}(s_k) - v_n(s_k)]ds_k \right)^2 ds_1 \cdots ds_{k-1}ds_{k+1} \cdots ds_i = 0$$

for $k = 1, 2, \cdots, i$, so that the proof of the full continuity of $Q(x, v)$ is now complete.

6. **A special case.** The deductions of §4 are therefore applicable to the integral equation

$$\phi(x) = \lambda \left[f(x) + \sum_{i=1}^{m} a_i \int \cdots \int K_i(x, s_1, \cdots, s_i) \right.$$

$$\left. \cdot \phi(s_1) \cdots \phi(s_i)ds_1 \cdots ds_i \right].$$

If we assume that $a_m=1$, $a_i=0$ for $i=1, 2, \cdots, m-1$, that is, if we consider

$$\phi(x) = \lambda\Big[f(x) + \int \cdots \int K_m(x, s_1, \cdots, s_m) \cdot \phi(s_1) \cdots \phi(s_m) ds_1 \cdots ds_m\Big],$$

we know that it has at least one solution, and that this solution may belong to a finite or an infinite eigenvalue. The homogeneous equation

$$\phi(x) = \lambda \int \cdots \int K_m(x, s_1, \cdots, s_m)\phi(s_1) \cdots \phi(s_m) ds_1 \cdots ds_m,$$

however, has always at least one finite eigenvalue. In this case namely (see (6))

$$J(v_n) = 2e_m \int G_m(x, v_n)v_n(x)dx,$$

so that

$$d_n = \frac{2}{m+1}\int G_m(x, \phi_n)\phi_n(x)dx = \frac{2}{m+1}\cdot\frac{N^2}{\lambda_n}$$

or

$$\lambda_n \cdot d_n = (2/(m+1))N^2.$$

But since the functional

$$\frac{1}{2(m+1)}J(v) = \int \cdots \int K_m(x, s_1, \cdots, s_m) \cdot v(s_1) \cdots v(s_m) dx ds_1 \cdots ds_m,$$

$K_m \not\equiv 0$, $v \in \overline{H}^2$, certainly has a minimum $\bar{d}$ differing from zero,

$$\bar{\lambda} = \lim_{n\to\infty} \lambda_n = \frac{2N^2}{\bar{d}(m+1)}$$

is finite.

CARLETON COLLEGE

THE SOLUTION OF THE DIFFERENTIAL EQUATION $(a^2\,\partial^2/\partial t^2-\Delta)(\partial^2/\partial t^2-\Delta)u=f(x, y, z, t)$ BY HADAMARD'S METHOD

J. P. KORMES

1. Introduction. Hadamard [1][1] generalized and extended Riemann's [2] method of solution of the initial value problem to any hyperbolic linear differential equation of the second order in any number of variables. The essential part of Hadamard's work is his discovery of the significance of an elementary solution which is a function of the geodetic distance determined by the characteristic manifold of the differential equation. Given the differential equation $L[u]=f$ and the initial data on a manifold C, let $\Gamma=0$ be the equation of the characteristic manifold of the differential equation and $M[u]$ the adjoint expression of $L[u]$. The first step is the determination of an elementary solution V, that is, a solution of $M[V]=0$ which is singular on $\Gamma=0$. To obtain the solution of the initial value problem, Hadamard considers the domain bounded by the characteristic "conoid" through a point P, and the initial manifold $C=0$. Since the elementary solution is singular on the characteristic conoid, in order to apply Green's formula Hadamard uses an interior domain $G_{\epsilon,\delta}$ which approaches the original domain as $\epsilon\to 0$ and $\delta\to 0$. For a fixed δ each term of Green's formula has then the following form:

$$B(\epsilon) = b + (1/\epsilon^{(n-3)/2})(b_0 + b_1\epsilon + \cdots + b_{(n-3)/2}\epsilon^{(n-3)/2}) + (\epsilon),$$

where (ϵ) tends to zero as $\epsilon\to 0$ and the b_i are independent of ϵ. The term b, the "finite part" of the integral, has the property to remain invariant under all transformations of the parameter ϵ. It follows from Green's formula that the sum of all such "finite parts" is equal to zero, which gives a relation for the function u. As we let the parameter δ approach zero we obtain the desired expression for $u(P)$. Hadamard, and independently Friedrichs [3], showed that in the case of an even number of variables each term of Green's formula will have an element containing $\log\epsilon$. The coefficients of $\log\epsilon$ have the property to remain invariant under all transformations of the parameter ϵ and it follows from Green's formula that the sum of all coefficients of $\log\epsilon$ is equal to zero, which gives the desired relation for the function u.

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

The purpose of this paper is to show by a significant example that this method can be extended to problems of a more complex type.

I wish to express my appreciation to Professor R. Courant who suggested the subject and the method of attack, as well as to Dr. C. De Prima and Dr. B. Friedman for their generous assistance.

2. Elementary solution, Green's formula and domain of integration. We consider the initial value problem:

$$(2.1)\qquad \begin{aligned} L[u] &\equiv (a^2\partial^2/\partial t^2 - \Delta)(\partial^2/\partial t^2 - \Delta)u = f(x, y, z, t); \qquad a > 1; \\ u(x, y, z, 0) &= u_t(x, y, z, 0) = u_{tt}(x, y, z, 0) \\ &= u_{ttt}(x, y, z, 0) = 0, \end{aligned}$$

where Δ is the Laplace operator: $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, and the function $f(x, y, z, t)$ is assumed to have continuous derivatives up to the fourth order. The characteristic manifold is given by:

$$[(\tau - t)^2 - r^2] \cdot [(\tau - t)^2/a^2 - r^2] = 0,$$

where $r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$. We introduce the functions Γ and Γ':

$$\Gamma = (\tau - t)^2 - r^2; \qquad \Gamma' = (\tau - t)^2/a^2 - r^2.$$

The characteristic manifold consists of two conical sheets, the exteror: $\Gamma = 0$, and the interior: $\Gamma' = 0$.

As the elementary solution V of the self-adjoint equation $L[V] = 0$ we take:

$$(2.2)\qquad V = -\frac{1}{2r}\log\frac{(\tau - t) - r}{(\tau - t) + r} + \frac{a}{2r}\log\frac{(\tau - t)/a - r}{(\tau - t)/a + r}.$$

In order to obtain Green's formula for equation (2.1) we consider:

$$Q[u, V] = a^2 u_{tt}V_{tt} - a^2\sum_i u_{it}V_{it} - \sum_i V_{it}u_{it} + \sum_{i,k} u_{ii}V_{kk}$$
$$(i, k = x, y, z).$$

The integration of $Q[u, V]$ over a four-dimensional domain G gives rise to two types of integrals:

$$\text{(a)}\quad \iiiint_G u_{it}V_{it}\,dx\,dy\,dz\,dt \quad \text{and} \quad \text{(b)}\quad \iiiint_G u_{ii}V_{kk}\,dx\,dy\,dz\,dt$$
$$(i, k = x, y, z, t).$$

Integrating by parts and collecting the results we obtain the following Green's formula:

$$\iiiint_G \{VL[u] - uL[V]\}\,dg = \iiint_S \left\{V\frac{\partial N[u]}{\partial s_2} - u\frac{\partial M[V]}{\partial s_1} - N[u]\frac{\partial V}{\partial s_2} + M[V]\frac{\partial u}{\partial s_1}\right\}do, \tag{2.3}$$

where:

$$M = a^2\partial^2/\partial t^2 - \Delta; \qquad N = \partial^2/\partial t^2 - \Delta$$

and where

$$\partial/\partial s_1 = t_\nu\partial/\partial t - x_\nu\partial/\partial x - y_\nu\partial/\partial y - z_\nu\partial/\partial z$$

and

$$\partial/\partial s_2 = a^2t_\nu\partial/\partial t - x_\nu\partial/\partial x - y_\nu\partial/\partial y - z_\nu\partial/\partial z$$

are the transversal differentiations, x_ν, y_ν, z_ν, t_ν being the direction coefficients of the outward normal to S. If in $Q[u, V]$ we interchange u and V we obtain a *second form* of Green's formula:

$$\iiiint_G \{VL[u] - uL[V]\}\,dg = \iiint_S \left\{V\frac{\partial M[u]}{\partial s_1} - u\frac{\partial N[V]}{\partial s_2} - M[u]\frac{\partial V}{\partial s_1} + N[V]\frac{\partial u}{\partial s_2}\right\}do. \tag{2.4}$$

For the elementary solution V (2.2) we find readily:

$$M[V] = 2(a^2 - 1)(\tau - t)/\Gamma^2; \quad N[V] = 2(a^2 - 1)(\tau - t)/a^2\Gamma'^2. \tag{2.5}$$

We consider the domain G bounded by the hyperplane $t=0$ and the exterior sheet $\Gamma=0$ through a point $P(\xi, \eta, \zeta, \tau)$. The domain G is characterized by the inequalities: $\Gamma \geqq 0$; $0 \leqq t \leqq \tau$. In accordance with Hadamard's method we truncate the domain G by a plane $t=\tau-\delta$ and obtain the domain G_δ for which $\Gamma \geqq 0$, $0 \leqq t \leqq \tau-\delta$, $\delta>0$.

Since Γ' vanishes in the interior of G_δ, the elementary solution has there a singularity. In order to enable us to apply Green's formula, we subdivide G_δ into the following domains:

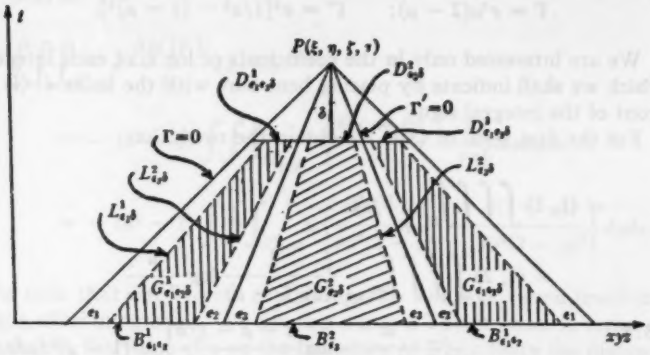
(a) the conical ring $G^1_{\epsilon_1\epsilon_2\delta}$:

$$0 \leqq t \leqq \tau - \delta;$$
$$(1 - \epsilon_1)^2(\tau - t)^2 - r^2 > 0; \qquad 0 < \epsilon_1 < 1 - 1/a,$$
$$(1/a + \epsilon_2)^2(\tau - t)^2 - r^2 < 0; \qquad 0 < \epsilon_2 < 1 - 1/a,$$

(b) the truncated interior cone $G^2_{\epsilon_3\delta}$:

$$(1 - \epsilon_3)^2((\tau - t)/a)^2 - r^2 > 0; \qquad 0 < \epsilon_3 < 1.$$

The ring domain is bounded by the volume $B^1_{\epsilon_1\epsilon_2}$, the lateral surfaces $L^1_{\epsilon_1\delta}$ and $L^2_{\epsilon_2\delta}$, and the volume $D^1_{\epsilon_1\epsilon_2\delta}$. Similarly the domain $G^2_{\epsilon_2\delta}$ is bounded by the sphere $B^2_{\epsilon_2}$, the lateral surface $L^2_{\epsilon_2}$, and the sphere $D^2_{\epsilon_2\delta}$ (see diagram).



3. **Integration over the domain $G^1_{\epsilon_1\epsilon_2\delta}$.** In $G^1_{\epsilon_1\epsilon_2\delta}$ where $(\tau-t)>r$ and $(\tau-t)/a<r$, the elementary solution is:

$$V = -\frac{1}{2r}\log\frac{(\tau-t)-r}{(\tau-t)+r}+\frac{a}{2r}\log\frac{r-(\tau-t)/a}{r+(\tau-t)/a}.$$

Since $L[V]=0$, Green's formula (2.3) may be written

$$-\iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} Vfdg+\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} Ado+\iiint_{L^1_{\epsilon_1}} Ado \tag{3.1}$$
$$+\iiint_{L^2_{\epsilon_2\delta}} Ado+\iiint_{B^1_{\epsilon_1\epsilon_2}} Ado=0$$

where

$$A = V\partial N[u]/\partial s_2 - u\partial M[V]/\partial s_1 - N[u]\partial V/\partial s_2 + M[V]\partial u/\partial s_1.$$

The last integral of (3.1) vanishes due to initial conditions and in order to express the remaining integrals as functions of ϵ and δ we introduce new coordinates, σ, μ; α, β, γ by the transformation

$$x=\xi+\sigma(1-\mu)\alpha, \qquad y=\eta+\sigma(1-\mu)\beta, \tag{3.2}$$
$$z=\zeta+\sigma(1-\mu)\gamma, \qquad t=\tau-\sigma,$$

where α, β, γ are parameters on the unit sphere $\alpha^2+\beta^2+\gamma^2=1$. The

domain $G^1_{\epsilon_1\epsilon_2\delta}$ is then determined by the inequalities $\delta \leqq \sigma \leqq \tau$ and $\epsilon_1 \leqq \mu \leqq 1-1/a-\epsilon_2$, and in the new coordinates:

$$\tau = \sigma(1-\mu); \quad dxdydzdt = \sigma^3(1-\mu)^2 d\mu d\sigma d\omega \quad [d\omega = (1/\gamma)d\alpha d\beta],$$
$$\Gamma = \sigma^2\mu(2-\mu); \quad \Gamma' = \sigma^2[1/a^2 - (1-\mu)^2].$$

We are interested only in the coefficients of $\log \epsilon_i$ of each integral which we shall indicate by placing bent bars with the index i: $\langle i \rangle$ in front of the integral sign.

For the first term of (3.1) we obtain the resolution

$$\begin{aligned}
&-\langle 1,2\rangle \iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} V f dg \\
&= \langle 1,2\rangle \iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} \left\{ \frac{1}{2\sigma(1-\mu)} \log \frac{\mu}{2-\mu} \right. \\
&\qquad \left. - \frac{a}{2\sigma(1-\mu)} \log \frac{1-\mu-1/a}{1-\mu+1/a} \right\} f\sigma^2(1-\mu)^2 d\mu d\sigma d\omega \qquad (3.3) \\
&= \frac{1}{2}\int_\delta^\tau \sigma^2 d\sigma \iint_\rho d\omega \langle 1,2\rangle \int_{\epsilon_1}^{1-1/a-\epsilon_2} \{(1-\mu)\log\mu \\
&\qquad - a(1-\mu)\log(1-\mu-1/a)\} f d\mu,
\end{aligned}$$

where ρ is the surface of the intersection of the conical ring $G^1_{\epsilon_1\epsilon_2\delta}$ with the plane $\sigma = \tau - t$. It can be readily seen that upon μ-integration the coefficients of $\log \epsilon_1$ and $\log \epsilon_2$ will have ϵ_1 and ϵ_2 respectively as factors and will therefore tend to zero as ϵ_1 and ϵ_2 approach zero. Therefore,

$$\lim_{\delta\to 0,\ \epsilon_1\to 0,\ \epsilon_2\to 0} \left(-\langle 1,2\rangle \iiiint_{G^1_{\epsilon_1\epsilon_2\delta}} V f dg \right) = 0. \qquad (3.4)$$

On $D^1_{\epsilon_1\epsilon_2\delta}: t_\nu = 1$, $x_\nu = y_\nu = z_\nu = 0$ and the second term of (3.1) can be written:

$$\begin{aligned}
&\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} \left\{ a^2 V \frac{\partial N[u]}{\partial t} - u \frac{\partial M[V]}{\partial t} \right. \\
&\qquad \left. - a^2 N[u] \frac{\partial V}{\partial t} + M[V] \frac{\partial u}{\partial t} \right\} dxdydz. \qquad (3.5)
\end{aligned}$$

We consider each of the four integrals of (3.5) separately. The first integral contains V and a resolution similar to that of (3.3) shows that the coefficients of $\log \epsilon_1$ and $\log \epsilon_2$ tend to zero as ϵ_1 and ϵ_2 approach zero. Observing that $M[V]=2(a^2-1)(\tau-t)/\Gamma^2$, the second integral of (3.5) is:

$$-\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} u\frac{\partial M[V]}{\partial t}\,dxdydz$$

$$=-(a^2-1)\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} u\left\{\frac{8(\tau-t)^2}{\Gamma^2}-\frac{2}{\Gamma^2}\right\}dxdydz$$

$$=-(a^2-1)\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} u\left\{\frac{8(1-\mu)^2}{\sigma\mu^2(2-\mu)^2}-\frac{2(1-\mu)^2}{\sigma\mu^2(2-\mu)^2}\right\}d\mu d\omega.$$

We note that only terms containing $1/\mu$ will give terms involving $\log \epsilon_i$ after integration. We develop u into a Taylor series in terms of $\hat{u}$, that is, the value of u on the boundary of $D^1_{\epsilon_1\epsilon_2\delta}$. Since the distance of a point $(\sigma, \mu, \alpha, \beta, \gamma)$ from the corresponding boundary point $(\sigma, \epsilon_1, \alpha, \beta, \gamma)$ is $\sigma(1-\epsilon_1)-\sigma(1-\mu)=\sigma(\mu-\epsilon_1)$, we have $u=\hat{u}-\sigma(\mu-\epsilon_1)\hat{u}_\nu+(\sigma^2(\mu-\epsilon_1)^2/2)\hat{u}_{\nu\nu}-\cdots=\hat{u}-\sigma\mu\hat{u}_\nu+(\sigma^2\mu^2/2)\hat{u}_{\nu\nu}-\cdots+(\epsilon_1)$, where (ϵ_1) denotes terms which tend to zero with ϵ_1 and $\hat{u}_\nu=\alpha\partial u/\partial x+\beta\partial u/\partial y+\gamma\partial u/\partial z$. Since the fourth and higher terms in the Taylor development of u contain $\sigma^2\mu^2$, the difference of u and the first three terms of its development will not give any coefficient of $\log \epsilon_i$. For the purpose of determining coefficients of $\log \epsilon_i$, we may write therefore

$$-\langle 1, 2\rangle\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} u\frac{\partial M[V]}{\partial t}\,dxdydz$$

$$=-\frac{a^2-1}{\sigma}\iint_\rho \hat{u}d\omega\ \langle 1, 2\rangle\int_{\epsilon_1}^{1-1/\sigma-\epsilon_2}\left\{\frac{8(1-\mu)^2}{\mu^3(2-\mu)^3}-\frac{2(1-\mu)^2}{\mu^2(2-\mu)^2}\right\}d\mu$$

$$+(a^2-1)\iint_\rho \hat{u}_\nu d\omega\ \langle 1, 2\rangle\int_{\epsilon_1}^{1-1/\sigma-\epsilon_2}\left\{\frac{8(1-\mu)^2}{\mu^2(2-\mu)^3}-\frac{2(1-\mu)^2}{\mu(2-\mu)^2}\right\}d\mu$$

$$-\frac{\sigma(a^2-1)}{2}\iint_\rho \hat{u}_{\nu\nu}d\omega\ \langle 1, 2\rangle\int_{\epsilon_1}^{1-1/\sigma-\epsilon_2}\left\{\frac{8(1-\mu)^2}{\mu(2-\mu)^3}-\frac{2(1-\mu)^2}{(2-\mu)^2}\right\}d\mu.$$

The coefficient of $\log \epsilon_1$ is zero in the first integral above, 1 in the

second and $-1/2$ in the third. Since the integration does not yield $\log \epsilon_2$ we have:

$$\lim_{\epsilon_1 \to 0,\ \epsilon_2 \to 0} \left(- \langle 1, 2 \rangle \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} u \frac{\partial M[V]}{\partial t} dx dy dz \right)$$

$$= (a^2 - 1) \iint_{\sigma \leq \delta} \bar{u}_r d\omega + \frac{\delta(a^2 - 1)}{2} \iint_{\sigma = \delta} \bar{u}_r d\omega.$$

But

$$\iint_{\sigma = \delta} \bar{u}_r d\omega = \frac{1}{\delta^2} \iint_{\sigma \leq \delta} \bar{u}_r d\Omega = \frac{1}{\delta^2} \iiint_{\tau = \delta} \Delta u dx dy dz,$$

where the last integral is an infinitesimal of the order δ^2 when δ approaches zero. We find therefore

$$(3.6) \quad \lim_{\delta \to 0} \lim_{\epsilon_1 \to 0,\ \epsilon_2 \to 0} \left\{ - \langle 1, 2 \rangle \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} u \frac{\partial M[V]}{\partial t} dx dy dz \right\} = 0.$$

The third integral of (3.5) is

$$- a^2 \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} N[u] \frac{\partial V}{\partial t} dx dy dz$$

$$= - a^2 \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} N[u] \left\{ \frac{1}{\Gamma} - \frac{1}{\Gamma'} \right\} dx dy dz$$

$$= - \sigma a^2 \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} N[u] \left\{ \frac{(1 - \mu)^2}{\mu(2 - \mu)} - \frac{(1 - \mu)^2}{[1/a^2 - (1 - \mu)^2]} \right\} d\mu d\omega$$

and as we let $\sigma = \delta$ approach zero, this integral vanishes.

The fourth integral of (3.5) is

$$\langle 1, 2 \rangle \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} M[V] u_t dx dy dz$$

$$= (a^2 - 1) \langle 1 \rangle \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} \frac{2(\tau - t)}{\Gamma^2} u_t dx dy dz$$

$$= (a^2 - 1) \langle 1 \rangle \iiint_{D^1_{\epsilon_1 \epsilon_2 \delta}} \frac{2 u_t (1 - \mu)^2}{\mu^2 (2 - \mu)^2} d\mu d\omega$$

$$= (a^2 - 1) \iint_{\sigma = \delta} \bar{u}_t d\omega \langle 1 \rangle \int_{\epsilon_1}^{1 - 1/a - \epsilon_2} \frac{2(1 - \mu)^2}{\mu^2 (2 - \mu)^2} d\mu$$

$$= \frac{a^2 - 1}{2} \iint_{\sigma = \delta} \bar{u}_t d\omega.$$

Whence

$$\lim_{\delta\to 0,\ \epsilon_1\to 0,\ \epsilon_2\to 0}\left\{\langle 1\rangle\iiint_{D^1_{\epsilon_1\epsilon_2\delta}} M[u]\frac{\partial u}{\partial t}\,dxdydz\right\} = 2\pi(a^2-1)u_t(\xi,\eta,\zeta,\tau). \tag{3.7}$$

On the lateral surface $L^1_{\epsilon_1\delta}$, μ is constant and since $M[V]$ and $\partial M[V]/\partial s_1$ contain no logarithms, we need only consider

$$\begin{aligned}\langle 1\rangle S^1_{\epsilon_1\delta} &= \langle 1\rangle\iiint_{L^1_{\epsilon_1\delta}}\left\{V\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial V}{\partial s_2}\right\}do\\ &= \langle 1\rangle\iiint_{L^1_{\epsilon_1\delta}}\left\{\left[-\frac{1}{2\sigma(1-\mu)}\log\frac{\mu}{2-\mu}\right.\right.\\ &\qquad\left.\left.+\frac{a}{2\sigma(1-\mu)}\log\frac{1-\mu-1/a}{1-\mu+1/a}\right]\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial V}{\partial s_2}\right\}do.\end{aligned}$$

On the lateral surface $(\partial V/\partial s_2)do=[(1-\epsilon_1)a^2V_t-\alpha V_x-\beta V_y-\gamma V_z]\cdot(1-\epsilon_1)^2\sigma^2 d\sigma d\omega$. Since $V_t=1/\Gamma-1/\Gamma'$, it does not contribute any coefficient of $\log\epsilon_1$,

$$V_x=\frac{\alpha}{2\sigma^2(1-\mu)^2}\log\frac{\mu}{2-\mu}-\frac{a\alpha}{2\sigma^2(1-\mu)^2}\log\frac{1-\mu-1/a}{1-\mu+1/a}+\cdots,$$

where the dots indicate terms which do not contain logarithms. Therefore

$$\begin{aligned}\langle 1\rangle\frac{\partial V}{\partial s_2}do &= \langle 1\rangle\left[-\frac{1}{2\sigma^2(1-\epsilon_1)^2}(\alpha^2+\beta^2+\gamma^2)\right](1-\epsilon_1)^2\sigma^2 d\sigma d\omega\\ &= -\,d\sigma d\omega/2.\end{aligned}$$

Thus the coefficient of $\log\epsilon_1$ for the lateral surface $L^1_{\epsilon_1\delta}$ as $\epsilon_1\to 0$ is

$$\lim_{\epsilon_1\to 0}(\langle 1\rangle S^1_{\epsilon_1\delta}) = -\frac{1}{2}\iiint_{L^1_\delta}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do.$$

In the limit $\delta\to 0$, we obtain

$$\lim_{\delta\to 0,\ \epsilon_1\to 0}(\langle 1\rangle S^1_{\epsilon_1\delta}) = -\frac{1}{2}\iiint_{L^1}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2}-N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do.^2 \tag{3.8}$$

[2] Since $\partial(1/r)/\partial s_2$ is not defined at the point $P(\xi,\eta,\zeta,\tau)$, this integral is to be taken in the limiting sense.

Similarly on the lateral surface $L^2_{\epsilon_2\delta}$ we need only consider

$$\langle 2\rangle S^2_{\epsilon_2\delta} = \langle 2\rangle \iiint_{L^2_{\epsilon_2\delta}} \left\{ V \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial V}{\partial s_2} \right\} do.$$

We have $\langle 2\rangle\ V = a/2r$; $\langle 2\rangle\ \partial V/\partial s_2 = (a/2)\partial(1/r)/\partial s_2$. Therefore the coefficient of $\log \epsilon_2$ for the lateral surface $L^2_{\epsilon_2\delta}$ is, when $\epsilon_2 \to 0$,

$$\lim_{\epsilon_2 \to 0} (\langle 2\rangle S^2_{\epsilon_2\delta}) = \frac{a}{2} \iiint_{L^2_\delta} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do.$$

In the limit $\delta \to 0$, we obtain:

$$(3.9) \quad \lim_{\delta \to 0} \lim_{\epsilon_2 \to 0} (\langle 2\rangle S^2_{\epsilon_2\delta}) = \frac{a}{2} \iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do$$

Summarizing the results obtained by the use of the first form of Green's formula (2.3), we have the equation

$$\left[2\pi(a^2 - 1) u_i(P) - \frac{1}{2} \iiint_{L^1} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do \right] \log \epsilon_1$$
$$+ \left[\frac{a}{2} \iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do \right] \log \epsilon_2 = 0.$$

Since ϵ_1 and ϵ_2 are independent, the coefficients of $\log \epsilon_1$ and $\log \epsilon_2$ must vanish, which gives the following relations:

$$(3.10) \quad 2\pi(a^2 - 1) u_i(P) = \frac{1}{2} \iiint_{L'} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do,$$

$$(3.11) \quad \frac{a}{2} \iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do = 0.$$

4. Application of the second form of Green's formula and integration over the domain $G^2_{\epsilon_2\delta}$. In a manner quite analogous to that in the preceding section we perform the integration of the second form of Green's formula (2.4) over the domains $G^1_{\epsilon_1\epsilon_2\delta}$ and $G^2_{\epsilon_2\delta}$ and obtain the following additional relations:

$$(4.1) \quad \frac{1}{2} \iiint_{L^1} \left\{ \frac{1}{r} \frac{\partial M[u]}{\partial s_1} - M[u] \frac{\partial (1/r)}{\partial s_1} \right\} do = 0,$$

$$(4.2) \quad 2\pi a(a^2 - 1) u_i(P) = -\frac{a}{2} \iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial M[u]}{\partial s_1} - M[u] \frac{\partial (1/r)}{\partial s_1} \right\} do.$$

Subtracting (4.2) from (3.10) we obtain an expression for u_t:

$$
\begin{aligned}
&2\pi(1-a)(a^2-1)u_t(P) \\
&\quad = \frac{1}{2}\iiint_{L^1}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do \\
&\quad + \frac{a}{2}\iiint_{L^2}\left\{\frac{1}{r}\frac{\partial M[u]}{\partial s_1} - M[u]\frac{\partial(1/r)}{\partial s_1}\right\}do.
\end{aligned} \tag{4.3}
$$

5. Transformation of lateral surface integrals into volume integrals. Let us consider the self-adjoint expression $M[u]=a^2\partial^2u/\partial t^2-\Delta u$. Green's formula for this expression of the second order is

$$
\begin{aligned}
&\iiiint_G \{VM[u] - uM[V]\}dg \\
&\quad = \iiint_S \left\{V\frac{\partial u}{\partial s_2} - u\frac{\partial V}{\partial s_2}\right\}do,
\end{aligned} \tag{5.1}
$$

where S is the three-dimensional boundary of G. If we put $V=1/r$ and $u=N[u]$, we cannot apply (5.1) over the interior of the surface $\Gamma=0$ because V has a singularity on the line $r=0$. To overcome this difficulty we again truncate the cone $\Gamma=0$ by the plane $t=\tau-\delta$ and remove the line $r=0$ by cutting out a small cone $\epsilon^2(\tau-t)^2-r^2=0$. Over this new domain, which we denote by $G^1_{\epsilon\delta}$, we apply formula (5.1):

$$
\begin{aligned}
&\iiiint_{G^1_{\epsilon\delta}}\left\{\frac{1}{r}M[N[u]] - N[u]M\left[\frac{1}{r}\right]\right\}dg \\
&\quad = \iiint_{S^1_{\epsilon\delta}}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do.
\end{aligned}
$$

Since $M[1/r]=0$ and, by (2.1), $M[N[u]]=f(x, y, z, t)$, we have

$$
\begin{aligned}
\iiiint_{G_\delta}\frac{1}{r}fdg &= \iiint_{L^1_\delta}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do \\
&+ \iiint_{D^1_{t\delta}}\left\{\frac{a^2}{r}\frac{\partial N[u]}{\partial t} - a^2N[u]\frac{\partial(1/r)}{\partial t}\right\}dxdydz \\
&+ \iiint_{L_{t\delta}}\left\{\frac{1}{r}\frac{\partial N[u]}{\partial s_2} - N[u]\frac{\partial(1/r)}{\partial s_2}\right\}do \\
&- \iiint_{B^1_t}\left\{\frac{a^2}{r}\frac{\partial N[u]}{\partial t} - a^2N[u]\frac{\partial(1/r)}{\partial t}\right\}dxdydz.
\end{aligned} \tag{5.2}
$$

The last integral of (5.2) vanishes because of initial conditions. Since

$$\iiint_{D^1_{\epsilon\delta}} \left\{ \frac{a^2}{r} \frac{\partial N[u]}{\partial t} - a^2 N[u] \frac{\partial (1/r)}{\partial t} \right\} dx dy dz$$

$$= \sigma^3 \iiint_{D^1_{\epsilon\delta}} \left\{ \frac{a^2}{r} \frac{\partial N[u]}{\partial t} - a^2 N[u] \frac{\partial (1/r)}{\partial t} \right\} (1-\mu)^2 d\mu d\omega,$$

the integral over $D^1_{\epsilon\delta}$ vanishes in the limit $\epsilon \to 0$, $\sigma = \delta \to 0$. In order to find the limiting value of the integral over $L_{\epsilon\delta}$, we make the following transformation of the coordinates:

$$x = \xi + \epsilon\sigma\alpha, \qquad y = \eta + \epsilon\sigma\beta, \qquad z = \zeta + \epsilon\sigma\gamma, \qquad t = \tau - \sigma.$$

In the new coordinates:

$$r = \epsilon\sigma, \quad do = \epsilon^2\sigma^2(1+\epsilon^2)^{1/2} d\sigma d\omega, \quad \partial(1/r)/\partial s_2 = -1/r^2(1+\epsilon^2)^{1/2},$$

and we obtain:

$$\iiint_{L_{\epsilon\delta}} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do$$

$$= \iiint_{L_{\epsilon\delta}} \left\{ \frac{1}{\epsilon\sigma} \frac{\partial N[u]}{\partial s_2} + N[u] \frac{1}{\epsilon^2\sigma^2(1+\epsilon^2)^{1/2}} \right\} \epsilon^2\sigma^2(1+\epsilon^2)^{1/2} d\sigma d\omega.$$

In the limit $\epsilon \to 0$, $\delta \to 0$ we obtain

$$\lim_{\epsilon \to 0,\ \delta \to 0} \left\{ \iint_{\tau - \delta\epsilon} d\omega \int_{\delta}^{\tau} N[u] d\sigma \right\} = -4\pi \int_0^{\tau} N[u] dt.$$

Therefore as we let ϵ and δ approach zero, (5.2) gives

$$\iiint_{G^1} \frac{1}{r} f dg = \iiint_{L^1} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do - 4\pi \int_0^{\tau} N[u] dt. \tag{5.3}$$

Interchanging $N[u]$ with $M[u]$ we obtain a similar expression, but, since by (4.1) the integral over L^1 vanishes, we have the relation

$$\iiint_{G^1} \frac{1}{r} f dg = -4\pi \int_0^{\tau} M[u] dt. \tag{5.4}$$

The same procedure applied to the domain G^2 consisting of the interior of the surface $\Gamma' = 0$ yields the following relations:

$$\iiint_{G^2} \frac{1}{r} f\,dg = \iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial M[u]}{\partial s_1} - M[u] \frac{\partial (1/r)}{\partial s_1} \right\} do - 4\pi \int_0^\tau M[u]\,dt, \tag{5.5}$$

$$\iiint_{G^2} \frac{1}{r} f\,dg = -4\pi \int_0^\tau N[u]\,dt. \tag{5.6}$$

6. Solution of the initial value problem. The relations of the preceding section enable us to obtain the solution of the initial value problem (2.1). Combining (5.3) and (5.6) we obtain

$$\iiint_{L^1} \left\{ \frac{1}{r} \frac{\partial N[u]}{\partial s_2} - N[u] \frac{\partial (1/r)}{\partial s_2} \right\} do = \iiint_{G^1} \frac{1}{r} f\,dg - \iiint_{G^2} \frac{1}{r} f\,dg, \tag{6.1}$$

and combining (5.4) and (5.5) gives the relation

$$\iiint_{L^2} \left\{ \frac{1}{r} \frac{\partial M[u]}{\partial s_1} - M[u] \frac{\partial (1/r)}{\partial s_1} \right\} do = \iiint_{G^2} \frac{1}{r} f\,dg - \iiint_{G^1} \frac{1}{r} f\,dg. \tag{6.2}$$

Substitution of the expressions (6.1) and (6.2) for the lateral surface integrals in (4.3) gives

$$4\pi(1-a)(a^2-1)u_i(P) = (1-a) \iiint_{G^1} \frac{1}{r} f\,dg - (1-a) \iiint_{G^2} \frac{1}{r} f\,dg$$

or

$$4\pi(a^2-1)u_i(P) = \iiint_{G^1} \frac{1}{r} f\,dg - \iiint_{G^2} \frac{1}{r} f\,dg. \tag{6.3}$$

Therefore the solution of the initial value problem (2.1) is

$$4\pi(a^2-1)u(\xi, \eta, \zeta, \tau) = \iiint_{G^1} \frac{\tau - t - r}{r} f\,dg - \iiint_{G^2} \frac{\tau - t - ar}{r} f\,dg. \tag{6.4}$$

In order to show that the right-hand side of (6.4) has the required derivatives we apply thereto transformation (3.2) and obtain

$$\iiiint_{G^3} f\sigma^3\mu(1-\mu)d\sigma d\mu d\omega - \iiiint_{G^3} f\sigma^3[1-a(1-\mu)](1-\mu)d\sigma d\mu d\omega.$$

Because of our assumptions concerning the function f both integrands are regular and our proof is complete.

7. **The limiting case** $a=1$. In order to determine what happens when $a=1$, we observe that the solution (6.4) for $a>1$ can be written in the form

$$\begin{aligned} 4\pi(a^2-1)u(P) &= \int_0^\tau dt \iiint_{r\leq\tau-t} \frac{\tau-t-r}{r} f\,dx\,dy\,dz \\ &\quad - \int_0^\tau dt \iiint_{r\leq(\tau-t)/a} \frac{\tau-t-ar}{r} f\,dx\,dy\,dz \\ &= F(1)-F(a). \end{aligned}$$

Therefore $4\pi(a+1)u(P)=-[F(1)-F(a)]/(1-a)$ and in the limit $a=1$, $8\pi u(P)=-F'(1)$. But differentiation of $F(a)$ with respect to a gives

$$F'(a) = -\int_0^\tau dt \iiint_{r\leq(\tau-t)/a} f\,dx\,dy\,dz,$$

whence

$$F'(1) = -\int_0^\tau dt \iiint_{r\leq\tau-t} f\,dx\,dy\,dz$$

and the solution of the iterated wave equation is

$$8\pi u(P) = \iiiint_{G^3} f\,dx\,dy\,dz\,dt. \tag{7.1}$$

8. **Nonhomogeneous initial conditions.** We consider now equation (2.1) with the following nonhomogeneous initial conditions:

$$\begin{aligned} &u(x,y,z,0)=\phi_0(x,y,z), \qquad u_t(x,y,z,0)=\phi_1(x,y,z), \\ &u_{tt}(x,y,z,0)=\phi_2(x,y,z), \quad u_{ttt}(x,y,z,0)=\phi_3(x,y,z). \end{aligned} \tag{8.1}$$

The introduction of the initial conditions (8.1) does not present anything new with regard to our method and we content ourselves in writing the solution which satisfies such conditions:

$$
\begin{aligned}
&4\pi(a^2-1)u(\xi,\eta,\zeta,\tau)\\
&\quad=\iiiint_{G^1}\frac{\tau-t-r}{r}f\,dx\,dy\,dz\,dt\\
&\quad-\iiiint_{G^2}\frac{\tau-t-ar}{r}f\,dx\,dy\,dz\,dt\\
&\quad+\iiint_{r\leqq\tau}\frac{1}{r}\Big\{(\tau-r)[a^2\phi_2-(a^2+1)\Delta\phi_1]\\
&\qquad\qquad+[a^2\phi_2-(a^2+1)\Delta\phi_0]\Big\}\,dx\,dy\,dz\\
&\quad-\iiint_{r\leqq\tau/a}\frac{1}{r}\Big\{(\tau-ar)[a^2\phi_2-(a^2+1)\Delta\phi_1]\\
&\qquad\qquad+[a^2\phi_2-(a^2+1)\Delta\phi_0]\Big\}\,dx\,dy\,dz\\
&\quad+\frac{a^2}{\tau}\iiint_{r\leqq\tau}\Delta\phi_0\,dx\,dy\,dz-\frac{a}{\tau}\iiint_{r\leqq\tau/a}\Delta\phi_0\,dx\,dy\,dz\\
&\quad+\frac{a^2}{\tau}\iint_{r=\tau}\left(\phi_1+\frac{1}{\tau}\phi_0\right)dS\\
&\quad-\frac{a^2}{\tau}\iint_{r=\tau/a}\left(\phi_1+\frac{1}{\tau}\phi_0\right)dS.
\end{aligned}
\tag{8.2}
$$

In the limiting case $a=1$, the above result reduces to

$$
\begin{aligned}
&8\pi u(\xi,\eta,\zeta,\tau)\\
&\quad=\iiiint_{G^1}f\,dg+\iiint_{r\leqq\tau}\left(\phi_3-\Delta\phi_1+\frac{2}{\tau}\Delta\phi_0\right)dx\,dy\,dz\\
&\quad+\iint_{r=\tau}\left(\phi_2+\frac{2}{\tau}\phi_1+\frac{2}{\tau^2}\phi_0-\Delta\phi_0\right)dS.
\end{aligned}
\tag{8.3}
$$

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ON UNIFORM CONVERGENCE OF TRIGONOMETRIC SERIES

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1. Introduction. The following theorems have been proved previously.[1]

THEOREM I. *If the function $\phi(t)$ is throughout continuous, periodic of period 2π, $\phi(t)=\phi(-t)=\phi(2\pi+t)$,*

$$\phi(t) \sim \frac{a_0}{2} + \sum_1^\infty a_n \cos nt, \tag{1.1}$$

and if

$$na_n > -K, \tag{1.2}$$

for some constant K, and all n, then the series (1.1) is uniformly convergent (on the real axis).

THEOREM II. *If $f(t)$ is everywhere continuous, periodic of period 2π, $f(t)=-f(-t)$,*

$$f(t) \sim \sum_1^\infty b_n \sin nt, \tag{1.3}$$

and if

$$nb_n > -K, \qquad n = 1, 2, 3, \cdots, \tag{1.4}$$

then the series (1.3) is uniformly convergent.

THEOREM III (CHAUNDY AND JOLLIFFE). *The Fourier series (1.3) is uniformly convergent, if*

$$b_n \geqq b_{n+1} > 0, \quad \text{and if} \quad nb_n \to 0. \tag{1.5}$$

Note that here no explicit assumption is made on $f(t)$.

THEOREM IV. *If $\phi(t)$ is continuous at $t=0$, and if*

$$\lim_{\lambda \downarrow 1} \limsup_{n\to\infty} \sum_n^{\lambda n} (|a_\nu| - a_\nu) = 0, \tag{1.6}$$

then the series (1.1) is uniformly convergent at $t=0$. (That is, $s_n(t_n)\to s$ whenever $t_n\to 0$.)

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[1] Cf. [2] and the references given there; numbers in brackets refer to the literature cited at the end of this paper.

THEOREM V. *If $f(t)$ is continuous at $t=0$, and if*

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} (|b_\nu| - b_\nu)^- = 0, \tag{1.7}$$

then $\sum_1^n \nu b_\nu = o(n)$, and the series (1.3) is uniformly convergent at $t=0$.

Some more general results are given in the present paper. In particular:

THEOREM 1. *Under the assumptions of Theorem* IV *the series* (1.1) *converges uniformly at each point of continuity of $\phi(t)$.*

THEOREM 2. *Under the assumptions of Theorem* V *the series* (1.3) *converges uniformly at each point of continuity of $f(t)$.*

Clearly Theorems 1 and 2 include Theorems I and II respectively. Either of the following two theorems includes Theorem III.

THEOREM 3. *Suppose that*

$$\sum_{n}^{2n} |b_\nu - b_{\nu+1}| = O(n^{-1}) \quad \text{as} \quad n \to \infty, \tag{1.8}$$

and that

$$(1-r)\sum_{1}^{\infty} n b_n r^n \to 0 \quad \text{as} \quad r \uparrow 1; \tag{1.9}$$

then the trigonometric series $\sum b_n \sin nt$ is uniformly convergent.

Note that the assumptions refer solely to the coefficients b_n.

THEOREM 4. *Suppose, for some constants $p \geqq 0$, $q \geqq 0$,*

$$nb_n + p = B_n \geqq 0, \tag{1.10}$$

that

$$B_{n+1} \leqq (1 + n^{-1}q)B_n, \quad \textit{for all large } n, \tag{1.11}$$

and that (1.9) *holds. Then $nb_n \to 0$ and the trigonometric series $\sum b_n \sin nt$ is uniformly convergent.*

We also give (in §§5 and 6) analogous theorems for cosine series; here the partial sums $\sum_1^n a_\nu = s_n$ play a similar role as the sequence $\{nb_n\}$ for the sine series. However convergence of the series $\sum a_\nu$ does not carry as far as existence of the limit $\lim nb_n$. It is for this reason that no such theorems have been established hitherto for cosine series. For details see §§5, 6 and 7.

2. Proof of Theorems 1 and 2. We have proved [2] that under the assumptions of Theorem IV

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} |a_\nu| = 0. \tag{2.1}$$

If $\phi(t)$ is continuous at t_0, then the Fourier series

$$\frac{\phi(t_0+\theta)+\phi(t_0-\theta)}{2} \sim \sum a_n \cos nt_0 \cos n\theta,$$

$$\frac{\phi(t_0+\theta)-\phi(t_0-\theta)}{2} \sim \sum a_n \sin nt_0 \sin n\theta$$

satisfy the assumptions of Theorems IV and V respectively, hence are uniformly convergent at $\theta=0$. This proves Theorem 1. The proof of Theorem 2 follows on quite similar lines, since it has been proved [2] that

$$\lim_{\lambda \downarrow 1} \limsup_{n \to \infty} \sum_{n}^{\lambda n} |b_\nu| = 0. \tag{2.2}$$

It is clear from our proof that the assumptions of our theorems can be replaced by the sole assumptions (2.1) and (2.2) respectively.

We remark that in Theorems IV and V the assumptions (1.6) and (1.7) cannot be replaced by

$$\sum_{n}^{2n} |a_\nu| = O(1) \quad \text{and} \quad \sum a_n \text{ converges},$$

$$\sum_{n}^{2n} |b_\nu| = O(1) \quad \text{and} \quad \sum_{1}^{n} \nu b_\nu = o(n),$$

respectively. We give an example, suggested by a construction due to Fejér [1].

Let $P_n(z)=\sum_{\nu=0}^{n-1} z^\nu/(n-\nu)-\sum_{\nu=0}^{n-1} z^{n+\nu}/(\nu+1)$, then $|P_n(z)|<6$ for $|z|\leqq 1$. Let $\mu_n=2^{n^2}$, $\kappa_n=2^{n(n+1)}$, $n=1, 2, 3, \cdots$, and consider the polynomial series $\sum_1^\infty n^{-2} z^{\mu_n} P_{\kappa_n}(ze^{it/n})$. This series is clearly uniformly convergent for $|z|\leqq 1$, the degree of the nth term is $2\kappa_n+\mu_n-1<\mu_{n+1}$, hence writing out the polynomials successively we get a power series, convergent for $|z|<1$: $\sum_1^\infty c_n z^n = F(z)$, and $F(e^{it})$ is the Fourier power series of a continuous function. The structure of P_n easily yields $\sum_n^{2n} |c_\nu| = O(1)$. It can be proved, as in Fejér's example, that the series $\sum c_n e^{int}$ converges for each t, uniformly in $\epsilon \leqq t \leqq 2\pi-\epsilon$, $\epsilon>0$; but neither component converges uniformly at $t=0$. The same is true for

the series $\sum a_n \cos nt$, $\sum a_n \sin nt$, where $a_n = R(c_n)$; $\sum a_n$ converges, so that $\sum_1^n \nu a_\nu = o(n)$. Again, using Fejér's device, and replacing $e^{it/n}$ by e^{it_n}, where the sequence $\{t_n\}$ is everywhere dense in $(0, 2\pi)$, we get a continuous function with a Fourier series and its conjugate nonuniformly convergent everywhere, while $|c_n|$ is the same as before.

3. **Proof of Theorem 3.** It follows from (1.8) that $\lim b_n$ exists, and now from (1.9) that $\lim b_n = 0$. Furthermore

$$\sum_1^{2^\kappa} |b_\nu - b_{\nu+1}| \leqq \sum_{n=0}^{\kappa-1} \sum_{2^n}^{2^{n+1}} |b_\nu - b_{\nu+1}| = \sum \frac{1}{2^n} O(1) = O(1),$$

hence

$$\sum_1^\infty |b_\nu - b_{\nu+1}| < \infty. \tag{3.1}$$

Moreover

$$\begin{aligned} \sum_n^\infty |b_\nu - b_{\nu+1}| &\leqq \sum_{\kappa=0}^\infty \sum_{n\cdot 2^\kappa}^{n\cdot 2^{\kappa+1}} |b_\nu - b_{\nu+1}| \\ &= O\left(\frac{1}{n} \sum \frac{1}{2^\kappa}\right) = O\left(\frac{1}{n}\right), \end{aligned} \tag{3.2}$$

hence

$$nb_n = n \sum_n^\infty (b_\nu - b_{\nu+1}) = O(1). \tag{3.3}$$

It was proved by Littlewood that boundedness of a sequence and Abel summability imply $(C, 1)$ summability; if we apply this to the sequence $\{nb_n\}$ it follows from (1.9) and (3.3) that

$$\sum_1^n \nu b_\nu = o(n). \tag{3.4}$$

Next, from Abel's formula

$$\sum_n^m b_\nu \sin \nu t = \sum_n^{m-1} (b_\nu - b_{\nu+1}) T_\nu(t) + b_m T_m(t) - b_n T_{n-1}(t), \tag{3.5}$$

where

$$T_n(t) = \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2},$$

hence in any interval $\epsilon \leqq t \leqq 2\pi - \epsilon$

$$\left|\sum_{n}^{m} b_\nu \sin \nu t\right| < \epsilon^{-1}\pi \sum_{n}^{\infty} |b_\nu - b_{\nu+1}| + 2\epsilon^{-1}\pi(|b_n| + |b_m|).$$

Thus the series $\sum b_n \sin nt$ is uniformly convergent in $\epsilon \leqq t \leqq 2\pi - \epsilon$, $\epsilon > 0$. Let

$$\sum_{1}^{\infty} b_n \sin nt = f(t);$$

we shall prove next that $f(t) \to 0$ as $t \downarrow 0$. We write

$$f(t) = \left(\sum_{1}^{n} + \sum_{n+1}^{\infty}\right) b_\nu \sin \nu t = U_1(t) + U_2(t),$$

say, where $n = [\epsilon^{-1}t^{-1}]$. Now, employing (3.2), (3.3) and (3.5)

$$|U_2(t)| < t^{-1}\pi \left(\sum_{n+1}^{\infty} |b_\nu - b_{\nu+1}| + |b_{n+1}|\right) \tag{3.6}$$
$$= t^{-1}O(n^{-1}) = \epsilon O(1).$$

As to $U_1(t)$, we have

$$U_1 = \sum_{1}^{n} \nu b_\nu \frac{\sin \nu t}{\nu} = \sum_{1}^{n-1} v_\nu \Delta_\nu + v_n \frac{\sin nt}{n},$$

where

$$v_n = \sum_{1}^{n} \nu b_\nu, \qquad \Delta_n = \Delta \frac{\sin nt}{n} = \frac{\sin nt}{n} - \frac{\sin (n+1)t}{n+1}.$$

We have

$$\Delta_n = \int_0^t (\Delta \cos nx)dx = R \int_0^t z^n(1 - z)dx, \qquad z = e^{ix},$$

hence

$$|\Delta_n| < \int_0^t |1 - z|\, dx < t^2,$$

and

$$|U_1(t)| < t^2 \sum_{1}^{n} |v_\nu| + n^{-1}|v_n| \tag{3.7}$$
$$< \epsilon^{-2}n^{-1} \sum_{1}^{n} \nu^{-1}|v_\nu| + n^{-1}|v_n| \to 0$$

as $t \downarrow 0$, by (3.4).

Now (3.6) and (3.7) yield

$$\limsup_{t\to 0} |f(t)| \leqq \epsilon;$$

ϵ being arbitrary, we get $f(t)\to 0$ as $t\to 0$. In view of (3.3) uniform convergence now follows from Theorem II.

We remark that under the assumptions of Theorem 3 the sequence $\{nb_n\}$ need not have a limit. This is seen from the example

$$nb_n = 1 \quad \text{for} \quad n = 2^\nu,\ \nu = 0, 1, 2, \cdots, \quad b_n = 0 \text{ otherwise.}$$

Moreover in this case $b_n \geqq 0$ and $\sum b_n$ is convergent.

On the other hand for the example $\sum_2^\infty (-1)^n \sin (2n-1)t/n \log n$, $nb_n\to 0$, $\sum b_n$ converges, yet the series is divergent for $t=\pi/2$. Of course (1.8) is not satisfied, but $\sum_n^{2n} |b_\nu| = O(1/\log n)$.

4. Proof of Theorem 4. We shall employ the following lemma.

LEMMA 1. *Suppose that* $B_n \geqq 0$, *that for some* $q \geqq 0$

$$B_{n+1} \leqq (1 + q/n)B_n, \qquad n = 1, 2, 3, \cdots, \tag{4.1}$$

and that the sequence $\{B_n\}$ *is Abel summable to* B; *then* $B_n\to B$.

This is Lemma 5 of my paper [2]. Note that the inequalities $B_n \geqq 0$ and (4.1) need only be satisfied for all large n, $n \geqq n_0$, say. For the sequence $B_n' = B_{n_0}$, $n = 1, 2, \cdots, n_0$, $B_n' = B_n$, $n > n_0$, satisfies the assumptions of the lemma, hence $\lim B_n = \lim B_n'$ exists.

Now for $nb_n + p = B_n$, from (1.9)

$$(1 - r)\sum B_n r^n \to p \quad \text{as} \quad r\uparrow 1; \tag{4.2}$$

from (1.10) and (1.11)

$$0 \leqq B_{n+1} \leqq (1 + q/n)B_n, \quad \text{for all large } n. \tag{4.3}$$

Lemma 1 now yields

$$B_n \to p, \quad \text{that is} \quad nb_n \to 0. \tag{4.4}$$

From (4.3)

$$(B_{n+1} - B_n) \leqq n^{-1}qB_n, \quad \text{for } n \geqq n_0, \text{ say.} \tag{4.5}$$

Write $\sum_n^{2n}(B_{\nu+1} - B_\nu) = \sum' + \sum''$, where $\sum'$ is the sum of the positive terms, and $\sum''$ the rest. From (4.4) and (4.5), $\sum' = O(1)$; furthermore

$$B_{2n+1} - B_n = \sum' + \sum'' = \sum' - |\sum''|,$$

hence $|\sum''| = B_n - B_{2n+1} + \sum' = O(1)$. It now follows that

$$\sum_{n}^{2n} |B_{\nu+1} - B_\nu| = \sum' + |\sum''| = O(1);$$

this and (4.4) yield (1.8). Our theorem now follows from Theorem 3.

If we replace (1.9) by the assumption (A) $\lim nb_n = \rho$, then the trigonometric series

$$\sum (b_n - \rho n^{-1}) \sin nt = \sum \beta_n \sin nt$$

satisfies the assumptions of Theorem 4, hence it is uniformly convergent, and we get $nb_n \to \rho$, and

$$\sum b_n \sin nt \to \pi\rho/2 \quad \text{as} \quad t \downarrow 0. \tag{4.6}$$

Combined with Theorem 3 of our paper [2] we get the theorem.

THEOREM 5. *If* (4.2) *holds then a necessary and sufficient condition that* (4.6) *holds is* $nb_n \to \rho$.

For b_n positive and decreasing, $\rho = 0$, the result is due to Chaundy and Jolliffe, for $\rho \neq 0$ to Hardy. For references see [2].

5. **The cosine series.** We shall next prove the theorem:

THEOREM 6. *Suppose that*

$$\sum_{n}^{2n} |a_\nu - a_{\nu+1}| = O(n^{-1}), \tag{5.1}$$

and that $\sum a_n$ *is Abel summable, then* $\sum a_n \cos nt$ *is uniformly convergent.*

Using Abel's formula

$$\sum_{n}^{m} a_\nu \cos \nu t = \sum_{n}^{m-1} (a_\nu - a_{\nu+1})\gamma_\nu(t) + a_m\gamma_m(t) - a_n\gamma_{n-1}(t). \tag{5.2}$$

where

$$\gamma_n(t) = \frac{\sin (n + 1/2)t}{2 \sin (t/2)}.$$

As in §3 it follows from (5.1) that $\lim a_n$ exists, and now Abel summability of $\sum a_n$ implies that $a_n \to 0$. Furthermore

$$\sum_{1}^{\infty} |a_n - a_{n+1}| < \infty, \quad \sum_{n}^{\infty} |a_\nu - a_{\nu+1}| = O(n^{-1}), \quad na_n = O(1). \tag{5.3}$$

Hence, by a theorem of Littlewood, $\sum a_n$ converges.

Now (5.2) yields uniform convergence of $\sum a_n \cos nt$ in $\epsilon \leqq t \leqq \pi$, $\epsilon > 0$. Let

$$\sum_1^\infty a_n \cos nt = \phi(t), \qquad 0 < t \leqq \pi.$$

We write

$$\sum_1^\infty a_\nu \cos \nu t = \sum_1^n + \sum_{n+1}^\infty = V_1(t) + V_2(t),$$

say, where $n = [\epsilon^{-1}t^{-1}]$. Now from (5.2)

$$V_2(t) = \sum_{n+1}^\infty (a_\nu - a_{\nu+1})\gamma_\nu(t) - a_{n+1}\gamma_n(t),$$

hence

$$|V_2(t)| < t^{-1}\pi\left(|a_{n+1}| + \sum_{n+1}^\infty |a_\nu - a_{\nu+1}|\right) \tag{5.4}$$

$$= t^{-1}O(n^{-1}) = \epsilon O(1).$$

To estimate V_1 put $\sum_n^\infty a_\nu = r_n$, then $r_1 = s$, and

$$V_1 = s\cos t - r_{n+1}\cos nt + 2\sum_2^n r_\nu \sin(t/2)\sin(\nu + 1/2)t,$$

hence

$$|V_1(t) - s\cos t| \leqq |r_{n+1}| + t\sum_2^n |r_\nu| \tag{5.5}$$

$$\leqq |r_{n+1}| + \epsilon^{-1}n^{-1}\sum_2^n |r_\nu| = \epsilon^{-1}o(1)$$

as $t \to 0$. From (5.4) and (5.5) $\limsup_{t\to 0} |\phi(t) - s| \leqq \epsilon$, ϵ being arbitrary, we get $\phi(t) \to s$, as $t \to 0$. Our theorem now follows from Theorem I. The example $\sum 2^{-n}\cos 2^n t$ shows that na_n need not have a limit.

Here is an alternative proof for the continuity of $\phi(t)$ at $t = 0$: From (5.2)

$$\phi(t) = -a_1/2 + 2^{-1}\sum_1^\infty (a_n - a_{n+1})\cos nt$$

$$+ 2^{-1}\cos(t/2)\sum_1^\infty (a_n - a_{n+1})\frac{\sin nt}{\sin(t/2)};$$

clearly $\sum (a_n - a_{n+1})\cos nt$ is uniformly convergent. Furthermore

$$\sum_1^\infty (a_n - a_{n+1})\sin nt = \sum_1^\infty n(a_n - a_{n+1})\frac{\sin nt}{n} = \sum_1^\infty a_n'\frac{\sin nt}{n},$$

where

$$a_n' = n(a_n - a_{n+1}), \qquad \sum_n^{2n} |a_\nu'| = O(1), \quad \text{by (5.1).}$$

Now $\sum_1^n a_\nu' = \sum_1^{n+1} a_\nu - (n+1)a_{n+1}$; $\sum a_n$ being convergent, it follows that $n^{-1}\sum_1^n \nu a_\nu \to 0$, and $\sum a_n'$ is $(C, 1)$ summable to s, hence by Theorem 4 of our paper [3]

$$\sum_1^\infty a_n' \frac{\sin nt}{nt} \to \sum_1^\infty a_n = s, \quad \text{as} \quad t \to 0.$$

Thus $\phi(t)$ is continuous at $t=0$.

Theorems 3 and 6 combined yield the theorem:

THEOREM 7. *Suppose that*

$$\sum_n^{2n} |c_\nu - c_{\nu+1}| = O(n^{-1}) \quad as \quad n \to \infty,$$

and that $\sum c_n$ is Abel summable; then the power series $\sum c_n z^n$ is uniformly convergent in the circle $|z| \leq 1$.

It suffices to consider the circle $|z| = 1$; suppose first that the c_n are real. The uniform convergence of $\sum c_n \cos nt$ follows from Theorem 6; it also follows that $n^{-1}\sum_1^n \nu c_\nu \to 0$, and Theorem 3 now yields the uniform convergence of $\sum c_n \sin nt$. If the c_n are complex, $c_n = a_n + ib_n$, then apply the result just obtained to $\sum a_n z^n$, $\sum b_n z^n$. This proves Theorem 7.

6. **Further theorems on cosine series.** Our next theorem is:

THEOREM 8. *Suppose that for some constants $p \geq 0$ and $q \geq 0$*

$$(6.1) \quad 0 \leq (n+1)s_{n+1} - ns_n + p \leq (1 + q/n)[ns_n - (n-1)s_{n-1} + p],$$

$s_n = \sum_1^n a_\nu$, and that $\sum a_n$ is Abel summable; then $na_n \to 0$, and $\sum a_n \cos nt$ is uniformly convergent.

Put $ns_n - (n-1)s_{n-1} + p = \delta_n = s_n + (n-1)a_n + p$, $s_0 = 0$, then $\sum_1^n \delta_\nu = n(s_n + p) \geq 0$, $s_n \geq -p$, hence by a well known theorem of Tauberian type $\sum a_n$ is $(C, 1)$ summable, thus the sequence $\{\delta_n\}$ is $(C, 2)$ summable. This and $0 \leq \delta_{n+1} \leq (1 + q/n)\delta_n$ imply by Lemma 1 that $\lim \delta_n$ exists, $\delta_n \to \delta$, say. It follows that $n^{-1}\sum_1^n \delta_\nu = s_n + p \to \delta$, or $s_n \to \delta - p = s$, and now

$$(6.2) \qquad na_n = \delta_n - s_n + a_n - p \to 0.$$

Next from (6.1)

$$\delta_{n+1} - \delta_n \leqq (q/n)\delta_n, \tag{6.3}$$

furthermore

$$D_n = \sum_n^{2n} (\delta_{\nu+1} - \delta_\nu) = \delta_{2n+1} - \delta_n = O(1). \tag{6.4}$$

Write $D_n = D' + D''$, where D' denotes the sum of positive terms, $D'' = D_n - D'$. From (6.3)

$$0 \leqq D' \leqq q \sum_n^{2n} \nu^{-1}\delta_\nu = O(1),$$

and now from (6.4), $|D''| = O(1)$, hence

$$\sum_n^{2n} |\delta_{\nu+1} - \delta_\nu| = O(1).$$

Also $\delta_{\nu+1} - \delta_\nu = (\nu+1)(a_{\nu+1} - a_\nu) + 2a_\nu$, thus

$$\sum_n^{2n} \nu |a_{\nu+1} - a_\nu| \leqq O(1) + 2\sum_n^{2n} |a_\nu|.$$

But from (6.2), $\sum_n^{2n} |a_\nu| = O(1)$, hence

$$\sum_n^{2n} |a_{\nu+1} - a_\nu| = O(n^{-1}).$$

Our theorem now follows from Theorem 6.

We next prove the following analogue to Lemma 1:

LEMMA 2. *Suppose that* $B_n \geqq 0$ *for* $n > n_0$, *that for some* $q > 0$

$$B_{n+1} \geqq (1 - q/n)B_n, \qquad n > n_0, \tag{6.5}$$

and that (A) $\lim B_n = B$; *then* $B_n \to B$.

We may assume that $q/n < 1$ for $n > n_0$; then from (6.5)

$$\sum_n^{n+\varepsilon} B_\nu \geqq B_n \sum_{\nu=0}^{\varepsilon} (1 - q/n)^\nu = \frac{nB_n}{q}\left\{1 - (1 - q/n)^{\varepsilon+1}\right\}, \quad n > n_0,$$

hence

$$B_n \leqq \frac{q}{n}(B'_{n+\varepsilon} - B'_{n-1})\left\{1 - (1 - q/n)^{\varepsilon+1}\right\}^{-1}, \text{ where}$$
$$B'_n = \sum_1^n B_\nu. \tag{6.6}$$

Choose $\kappa=[\delta n]$, where $\delta>0$; from Abel summability and from $B_n\geqq 0$, it follows that $n^{-1}B_n'\to B$. Now from (6.6)

$$\limsup B_n \leqq \frac{q\delta B}{1-\exp(-q\delta)};$$

letting $\delta\downarrow 0$, we get

$$\limsup B_n \leqq B.$$

Similarly for $n-\kappa>n_0$

$$\sum_{\nu=0}^{\varepsilon} B_{n-\nu} \leqq B_n \sum_{\nu=0}^{\varepsilon}\left(1-\frac{q}{n-\kappa}\right)^{-\nu}$$
$$=\frac{n-\kappa}{q}B_n\left(1-\frac{q}{n-\kappa}\right)\left\{\left(1-\frac{q}{n-\kappa}\right)^{-(\varepsilon+1)}-1\right\},$$

hence

$$B_n \geqq (q/(n-\kappa-q))(B_n'-B_{n-\kappa-1}')\left\{(1-q/(n-\kappa))^{-(\varepsilon+1)}-1\right\}^{-1}.$$

Let now $\kappa=[n\delta]$, $0<\delta<1$, then

$$\liminf B_n \geqq \frac{q\delta}{1-\delta}B\left(\exp\frac{q\delta}{1-\delta}-1\right)^{-1},$$

and $\delta\downarrow 0$ yields $\liminf B_n\geqq B$. This proves the lemma.

THEOREM 9. *Suppose that for some constants* $p\geqq 0$, $q\geqq 0$,

$$(6.7)\qquad (n+1)s_{n+1}-ns_n+p \geqq (1-q/n)[ns_n-(n-1)s_{n-1}+p]\geqq 0,$$

and that (A) $\lim s_n=s$ *exists. Then* $na_n\to 0$ *and* $\sum a_n\cos nt$ *is uniformly convergent.*

As in the proof of Theorem 8, $s_n\geqq -p$, hence $\sum a_n$ is $(C,1)$ summable; then by Lemma 2, $\delta_n\to\delta$, $na_n\to 0$. Next from (6.7)

$$\delta_{n+1}-\delta_n \geqq -qn^{-1}\delta_n, \quad \text{and}$$

$$(6.8)\qquad D_n=\sum_{n}^{2n}(\delta_{\nu+1}-\delta_\nu)=\delta_{2n+1}-\delta_n=O(1).$$

Write $D_n=D'+D''$, where D' denotes the sum of negative terms, $D''=D_n-D'$. From (6.8), $0\geqq D'\geqq -q\sum_n^{2n}\nu^{-1}\delta_\nu$, hence $D'=O(1)$, and $D''=O(1)$; hence $\sum_n^{2n}|\delta_{\nu+1}-\delta_\nu|=O(1)$. The remaining part is the same as in the proof of Theorem 8.

7. Closing remarks. The assumption of Lemma 1 can be written as $0 \leqq B_{n+1} \leqq (n+q)/nB_n$, or $0 \leqq (\Gamma(n+q)/\Gamma(n))B_{n+1} \leqq (\Gamma(n+q+1)/\Gamma(n+1))B_n$, that is, $\Gamma(n)B_n/\Gamma(n+q)$ is decreasing. A similar lemma was proved by Hardy; for reference see [2]. Again in Lemma 2 the assumption is $B_{n+1} \geqq (n-q)/nB_n \geqq 0$, or

$$(\Gamma(n-q)/\Gamma(n))B_{n+1} \geqq (\Gamma(n-q+1)/\Gamma(n+1))B_n \geqq 0,$$

that is, $\Gamma(n)B_n/\Gamma(n-q)$ is increasing. The larger q the more general is the condition.

The differences $(n+1)s_{n+1}-ns_n=\tau_{n+1}$ are the $(C, -1)$ means of the series $\sum a_n$, that is, $s_n=n^{-1}\sum_1^n \tau_\nu$ $(\tau_1=s_1)$. The condition (6.1) may be written as

$$-(\tau_n+p) \leqq \tau_{n+1}-\tau_n \leqq (q/n)(\tau_n+p).$$

If it holds for some p, then it clearly holds for any $p'>p$. Similarly (6.7) becomes

$$\tau_{n+1}-\tau_n \geqq -(q/n)(\tau_n+p) \geqq -(\tau_n+p),$$

and here too p may be replaced by any $p'>p$. Clearly summability $(C, -1)$ of the series $\sum a_n$ is equivalent to convergence together with $na_n \to 0$.

We have seen that the first inequality of (6.1) and Abel summability of $\sum a_n$ imply $(C, 2)$ summability of the sequence $\{\tau_n\}$; it follows from a theorem of Tauberian type that $\sum a_n$ converges. It is an open question whether this and $\tau_n \geqq -p$, $n=1, 2, 3, \cdots$, imply uniform convergence of $\sum a_n \cos nt$ at $t=0$. Theorem IV asserts that this is the case if $\sum a_n \cos nt$ is the Fourier series of a function continuous at $t=0$. However it is doubtful whether even $(C, -1)$ summability of $\sum a_n$ itself implies uniform convergence of $\sum a_n \cos nt$, or continuity of the corresponding function at $t=0$.

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A GENERALIZATION OF CONTINUED FRACTIONS[1]

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1. **Introduction.**[2] The generalizations and analogues of regular continued fractions due to Pierce [8], Lehmer [5], and Leighton [6] concern the iteration of rational functions to obtain rational approximations to a real number. The present generalization proceeds from the fact that the continued fraction

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\cdots}} \tag{1.1}$$

can be written in the form

$$f(a_1+f(a_2+\cdots \tag{1.2}$$

where $f(t)=1/t$. This suggests the possibility of using functions other than $1/t$ to obtain generalizations of (1.1). In §2 a class F of functions which includes $1/t$ is defined and in §3 meaning is given to (1.2) for each $f\in F$ and each sequence $a_1, a_2, a_3, \cdots$ of positive integers. An algorithm is given for obtaining for a fixed $f\in F$ an expression of the form (1.2) corresponding to each number x in the interval $0<x<1$; this expression is then called the *f-expansion of* x. The analogue of the nth convergent of a simple continued fraction is defined, and its behavior with respect to x is noted. In §4 the form (1.2) is called an *f-expansion* when $f\in F$ and $a_1, a_2, a_3, \cdots$ is a sequence of positive integers. The convergence and some idea of the rapidity of convergence of an f-expansion are established. The one-to-one correspondence between f-expansions and f-expansions of numbers x, $0<x<1$, is given in §5 by Theorem 5. In §6 statistical independence of the a_i of an f-expansion is defined in the customary way and a subclass F_p of F for which the a_i are statistically independent is considered. Various sets of numbers x whose f-expansions are restricted by conditions on the a_i are considered and the linear Lebesgue measures of these sets are given. In §7, when $f\in F_p$, certain sets of numbers x which have been studied for $f(t)=1/t$ by Borel [2] and F. Bernstein [1] are shown to be of measure zero.

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[2] Numbers in brackets refer to the bibliography.

2. The class F. Let F denote the class of real functions $f(t)$ defined for $t \geqq 1$ and having the following properties:

$$f(1) = 1; \tag{2.1}$$

$$f(t_1) > f(t_2) > 0, \qquad 1 \leqq t_1 < t_2; \tag{2.2}$$

$$\lim_{t \to \infty} f(t) = 0; \tag{2.3}$$

$$|f(t_2) - f(t_1)| < |t_2 - t_1|, \qquad 1 \leqq t_1 < t_2; \tag{2.4}$$

there is a constant λ such that $0 < \lambda < 1$ and

$$|f(t_2) - f(t_1)| < \lambda^2 |t_2 - t_1|, \qquad 1 + f(2) < t_1 < t_2. \tag{2.5}$$

3. The f-expansions of numbers. Let $f(t) \in F$ and x be a fixed number, $0 < x < 1$. Let z_0 be defined by $x = f(z_0)$ and let the sequences $z_1, z_2, \cdots, \theta_1, \theta_2, \cdots$, and $a_1, a_2, \cdots$ be defined by the relations

$$a_n = [z_{n-1}], \qquad \theta_n = z_{n-1} - a_n, \qquad \theta_n = f(z_n), \tag{3.1}$$

for $n = 1, 2, \cdots$. If $\theta_n \neq 0$ for $n < k$ while $\theta_k = 0$, we shall say the expansion terminates and that the *f-expansion of x* is[3]

$$f(a_1 + f(a_2 + \cdots + f(a_k). \tag{3.2}$$

In this case it is easy to see that $a_k \geqq 2$ and that the f-expansion of x is equal to x. If $\theta_n \neq 0$ for $n = 1, 2, \cdots$, then the expansion will not terminate and we shall call

$$f(a_1 + f(a_2 + \cdots \tag{3.3}$$

the *f-expansion of x*.

By analogy with simple continued fractions we define

$$x_n = f(a_1 + f(a_2 + \cdots + f(a_n) \tag{3.4}$$

and call the elements of the sequence $x_1, x_2, \cdots$ the *convergents* of x. The integers $a_1, a_2, \cdots$ and the convergents $x_1, x_2, \cdots$ are uniquely determined by x for almost all x, $0 < x < 1$. When we wish to emphasize this functional dependence we shall write them in the form $a_1(x), a_2(x), \cdots$ and $x_1(x), x_2(x), \cdots$.

To facilitate notation we introduce the function $\phi_n(t)$ defined when $f \in F$ and $a_1, a_2, \cdots$ is a sequence of positive integers by

$$\phi_n(t) = f(a_1 + f(a_2 + \cdots + f(a_n + t), \qquad t \geqq 0. \tag{3.5}$$

A simple induction proves the following lemma.

[3] In (3.2) and similar expressions we shall use a single parenthesis on the right.

LEMMA 1. *The function $\phi_n(t)$ is a decreasing (increasing) function of t when n is odd (even).*

THEOREM 1. *If $f \in F$ and $0<x<1$, then the odd (even) convergents of the f-expansion of x form a decreasing (increasing) sequence bounded below (above) by x; thus*

$$0 < x_2 < x_4 < \cdots \leqq x \leqq \cdots < x_3 < x_1 \leqq 1. \tag{3.6}$$

When $\phi_n(t)$ is defined by (3.5), we have $x_n=\phi_n(0)$, $x=\phi_n(\theta_n)$, and $x_{n+1}=\phi_n(f(a_{n+1}))$. Since $f(a_{n+1}) \geqq \theta_n > 0$, we can apply Lemma 1 to obtain $x_n > x \geqq x_{n+1}$ when n is odd and $x_n < x \leqq x_{n+1}$ when n is even. Since $f(a_{n+1}+f(a_{n+2})) > 0$ and $x_{n+2}=\phi_n(f(a_{n+1}+f(a_{n+2})))$, we similarly have $x_n > x_{n+2}$ when n is odd and $x_n < x_{n+2}$ when n is even. These inequalities establish Theorem 1.

COROLLARY. *If $\lim_{n\to\infty} x_n$ exists, then $\lim_{n\to\infty} x_n = x$.*

4. **Convergence of f-expansions.** If $f \in F$ we shall mean by an *f-expansion* either a finite expansion $f(a_1+f(a_2+\cdots+f(a_k)$ in which the a_i are positive integers and $a_k \geqq 2$, or an infinite expansion $f(a_1+f(a_2+\cdots$ in which the a_i are positive integers. It is to be proved later that each f-expansion is generated by a unique x; meanwhile this is not assumed.

THEOREM 2. *Let $f \in F$. If sequences x_n and y_n are defined in terms of an f-expansion by the formulas*

$$x_n = f(a_1 + f(a_2 + \cdots + f(a_n), \tag{4.1}$$

$$y_n = f(a_1 + f(a_2 + \cdots + f(a_n + 1), \tag{4.2}$$

then

$$0 < x_2 < x_4 < \cdots < x_3 < x_1 \leqq 1 \tag{4.3}$$

and

$$x_{n+1} \in I(a_1, a_2, \cdots, a_n), \tag{4.4}$$

where $I(a_1, a_2, \cdots, a_n)$ is the closed interval with end points at x_n and y_n.

Proof of (4.3) is identical with a part of the proof of (3.6). The conclusion (4.4) follows from Lemma 1 since $x_n=\phi_n(0)$,

$$x_{n+1} = \phi_n(f(a_{n+1})), \qquad y_n = \phi_n(1), \quad \text{and} \quad 0 < f(a_{n+1}) \leqq 1.$$

LEMMA 2.[4] *Let $f \in F$. For a fixed positive integer n, the least upper*

[4] We use the symbol $|E|$ to denote the linear Lebesgue measure of a set E.

bound of $|I(a_1, a_2, \cdots, a_n)|$ *for all sequences of positive integers* a_i *is less than* λ^{n-2} *where* λ *is the constant in* (2.5); *that is, if* $f \in F$ *and*

$$(4.5)\quad A_n = \text{l.u.b.}_{a_1, \cdots, a_n \geqq 1} |f(a_1 + \cdots + f(a_n + 1) - f(a_1 + \cdots + f(a_n)|,$$

where $a_1, a_2, \cdots, a_n$ *assume independently all positive integral values, then*

$$(4.6)\qquad A_n \leqq \lambda^{n-2}, \qquad n = 1, 2, \cdots.$$

For $n \geqq 1$, we can write

$$A_{n+2} = \text{l.u.b.}_{a_1, \cdots, a_{n+2} \geqq 1} \frac{|I(a_1, \cdots, a_{n+2})|}{|I(a_2, \cdots, a_{n+2})|} \cdot |I(a_2, \cdots, a_{n+2})|$$

$$\leqq A_n \cdot \text{l.u.b.}_{a_1, a_2 \geqq 1;\, 0<u<v\leqq 1} \left| \frac{f(a_1 + f(a_2 + u) - f(a_1 + f(a_2 + v)}{u - v} \right|,$$

from which we obtain

$$(4.7)\quad A_{n+2} \leqq A_n \cdot \text{l.u.b.}_{a_1, a_2 \geqq 1;\, 0<u<v\leqq 1} \left| \frac{f(a_1 + f(a_2 + u) - f(a_1 + f(a_2 + v)}{[a_1 + f(a_2 + u)] - [a_1 + f(a_2 + v)]} \right| \cdot \left| \frac{f(a_2 + u) - f(a_2 + v)}{u - v} \right|.$$

If $a_2 = 1$, then $a_1 + f(a_2 + u) > a_1 + f(a_2 + v) \geqq 1 + f(2)$ when a_1 is a positive integer and $0 < u < v \leqq 1$, so that by (2.5) and (2.4) the first and second factors of the product of which the least upper bound is taken in (4.7) are less than λ^2 and 1, respectively. If $a_2 \geqq 2 > 1 + f(2)$, then the first and second factors are less than 1 and λ^2, respectively. So we have $A_{n+2} \leqq \lambda^2 A_n$, $n = 1, 2, \cdots$. Since $A_2 \leqq A_1 < 1$, the statement (4.6) follows easily by mathematical induction.

THEOREM 3. *If* $f \in F$, *then each infinite f-expansion converges to a number* x *in the interval* $0 < x < 1$; *moreover*

$$(4.8)\qquad |x_n - x| \leqq \lambda^{n-2}, \qquad n = 1, 2, \cdots,$$

where λ *is the constant in* (2.5).

From Theorem 2 and Lemma 2 we conclude that $|x_{n+1} - x_n| \leqq \lambda^{n-2}$ for $n = 1, 2, \cdots$ and since $0 < \lambda < 1$, x_n converges to a number x which by (4.3) lies in each of the intervals from x_n to x_{n+1}. This proves (4.8).

THEOREM 4. *If* $f \in F$ *and* $0 < x < 1$, *the f-expansion of* x *converges to* x.

In the terminating case the f-expansion of x obviously equals x

and in this sense converges to x. In the non-terminating case the conclusion follows directly from Theorem 3 and the corollary to Theorem 1.

Henceforth we shall use the notation $x=f(a_1+f(a_2+\cdots$ to mean that the f-expansion on the right side converges to x.

When $f(t)=1/t$, the least upper bound of $|f(x)-f(y)|/|x-y|$ for $3/2<x<y$ is $(2/3)^2$, and so we may take $\lambda=2/3$. It follows from (4.8) that

$$|x_n(x)-x| \leqq (2/3)^{n-2}, \qquad n=1, 2, \cdots. \tag{4.9}$$

From the theory of simple continued fractions we know [7, 4] that

$$|x_n(x)-x| \leqq z^{n-1}, \qquad n=1, 2, \cdots, \tag{4.10}$$

where $z=(3-5^{1/2})/2$. Comparison of (4.9) and (4.10) shows that our method of obtaining estimates of the rapidity of uniform convergence of f-expansions gives, when applied to $f(t)=1/t$, an estimate which is similar in form to the stronger estimate of (4.10).

5. **Uniqueness.** In this section we establish a one-to-one correspondence between f-expansions and f-expansions of numbers x, $0<x<1$. We note, as in simple continued fractions [7, p. 22], the following lemma.

LEMMA 3. *If $f\in F$, then any two of the three equations*

$$x=f(a_1+f(a_2+\cdots, \tag{5.1}$$

$$y=f(a_n+f(a_{n+1}+\cdots, \tag{5.2}$$

$$x=f(a_1+f(a_2+\cdots+f(a_{n-1}+y) \tag{5.3}$$

implies the third, the f-expansions in (5.1) *and* (5.2) *being infinite.*

The proof of Lemma 3 is straightforward.

THEOREM 5. *If $f\in F$ and $0<x<1$, then an f-expansion which converges to x and the f-expansion of x are identical.*

If the two infinite f-expansions $f(a_1+f(a_2+\cdots$ and $f(b_1+f(b_2+\cdots$ converge to the same x, then by successively applying Lemma 3 we obtain $a_n=b_n$, $n=1, 2, \cdots$. A similar argument proves that an infinite f-expansion and a finite f-expansion or two different finite f-expansions do not converge to the same x. Theorem 4 completes the proof.

6. **Statistical independence.** From (3.6) and (4.4) we see that $I(c_1, c_2, \cdots, c_i)$ except for at most its end points is identical with the

set of x, $0<x<1$, for which $a_j(x)=c_j$, $j=1, 2, \cdots, i$. More exactly we have[5]

$$E[a_j(x) = c_j; j = 1, 2, \cdots, i] = I(c_1, c_2, \cdots, c_i) - \{f(c_1 + f(c_2 + \cdots + f(c_i + 1)\} \tag{6.1}$$

unless $i=1$ and $c_1=1$ in which case

$$E[a_1(x) = 1] = I(1) - \{f(1)\} - \{f(2)\}. \tag{6.2}$$

LEMMA 4. *If $f\in F$ and $c_1, c_2, \cdots, c_n$ and $c_1', c_2', \cdots, c_n'$ are two sets of positive integers such that for at least one j, $1\leqq j\leqq n$, $c_j\neq c_j'$, then the intervals $I(c_1, c_2, \cdots, c_n)$ and $I(c_1', c_2', \cdots, c_n')$ have at most an end point in common.*

The proof of this lemma follows from (6.1) and (6.2) and from the fact that the sets $E[a_j(x)=c_j;\ j=1, 2, \cdots, n]$ and $E[a_j(x)=c_j';\ j=1, 2, \cdots, n]$ are mutually exclusive by Theorem 5.

COROLLARY. *If $f\in F$, then*

$$I(c_1, c_2, \cdots, c_n) = \{f(c_1 + \cdots + f(c_n)\} + \sum_{j=1}^{\infty} I(c_1, c_2, \cdots, c_n, j).$$

If $y_1, y_2, \cdots$ is a decreasing sequence of positive numbers such that $y_1=1$ and $y_n\to 0$ and $f(t)$ is the function whose graph is the polygon joining in order the points (n, y_n), $n=1, 2, \cdots$, then $f(t)\in F$. Let F_p be the class of all such polygonal functions.

THEOREM 6. *If $f\in F_p$, then for any positive integers i and k*

$$|E[a_i(x) = k]| = f(k) - f(k+1).$$

By (6.1) and (6.2) we have $|E[a_1(x)=k]| = |I(k)| = f(k)-f(k+1)$. For any positive integer m, it follows from (6.1) and Lemma 4 that $|E[a_{m+1}(x)=k]| = \sum |I(b_1, b_2, \cdots, b_m, k)|$ where $\sum$ is to be taken independently over all positive integral values of $b_1, b_2, \cdots, b_m$. By the mean value theorem we have

$$\begin{aligned}
&|E[a_{m+1}(x) = k]| \\
&= \sum |f(b_1 + \cdots + f(k+1) - f(b_1 + \cdots + f(k)| \\
&= \sum |f(b_1) - f(b_1 + 1)|\,|f(b_2 + \cdots + f(k+1) - f(b_2 + \cdots + f(k)| \\
&= (\sum |f(b_1) - f(b_1 + 1)|)\cdot(\sum |I(b_2, \cdots, b_m, k)|) \\
&= \sum |I(b_2, \cdots, b_m, k)| = |E[a_m(x) = k]|.
\end{aligned}$$

[5] The symbol $E[\cdots]$ shall denote the set of x satisfying the proposition in brackets.

An induction completes the proof.

The functions $a_i(x)$, $i=1, 2, \cdots$, are said to be *statistically independent* [4] if for each set of positive integers $n_1<n_2<\cdots<n_m$ and each set of positive integers $c_1, c_2, \cdots, c_m$

$$(6.3)\quad \left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m] \right| = \prod_{j=1}^{m} \left| E[a_{n_j}(x) = c_j] \right|.$$

THEOREM 7. *If $f\in F_p$, then the functions $a_i(x)$, $i=1, 2, \cdots$, are statistically independent.*

The equation (6.3) is trivial for $m=1$. By (6.1) and Lemma 4 we have

$$\left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m] \right| = \sum{}' \left| I(b_1, \cdots, b_{n_1-1}, c_1 b_{n_1+1}, \cdots, c_2, \cdots, c_m) \right|$$

where $\sum'$ is to be taken independently over all positive integral values of b_i for all indices i from one to n_m excepting $i=n_1, n_2, \cdots, n_m$. By an argument similar to that used in the proof of Theorem 6 we obtain

$$\begin{aligned}
&\left| E[a_{n_j}(x) = c_j; j = 1, 2, \cdots, m] \right| \\
&\quad = \sum{}' \left| f(b_1) - f(b_1 + 1) \right| \cdot \left| I(b_2, \cdots, b_{n_1-1}, c_1, \cdots, c_m) \right| \\
&\quad = \sum{}' \left| I(b_2, \cdots, b_{n_1-1}, c_1, \cdots, c_m) \right| = \cdots \\
&\quad = \sum{}' \left| I(c_1, b_{n_1+1}, \cdots, c_m) \right| \\
&\quad = \left| f(c_1) - f(c_1 + 1) \right| \cdot \left(\sum{}' \left| I(b_{n_1+1}, \cdots, c_m) \right| \right) \\
&\quad = \left| E[a_{n_1}(x) = c_1] \right| \cdot \left| E[a_{n_j}(x) = c_j; j = 2, \cdots, m] \right|
\end{aligned}$$

and again an induction completes the proof.

COROLLARY. *If $f\in F_p$, then for each set of positive integers $n_1<n_2<\cdots<n_m$ and each set of positive integers $c_1, c_2, \cdots, c_m$, $d_1, d_2, \cdots, d_m$ such that $c_j\leqq d_j$, $j=1, 2, \cdots, m$, we have*

$$\begin{aligned}
\left| E[c_j \leqq a_{n_j}(x) \leqq d_j; j = 1, 2, \cdots, m] \right| &= \prod_{j=1}^{m} \left| E[c_j \leqq a_{n_j}(x) \leqq d_j] \right| \\
&= \prod_{j=1}^{m} \left| f(c_j) - f(d_j + 1) \right|.
\end{aligned}$$

7. Sets of measure zero.[6] The results of §6 will now be used in order to prove a few measuretheoretical facts concerning f-expansions under the assumption that $f\in F_p$.

[6] Theorems, similar to those in this section, applying to the simple continued fraction have been proved by Borel [2] and Bernstein [1]; for expositions see [3].

THEOREM 8. *If $f \in F_p$, then the set of x, $0<x<1$, for which the sequence $a_1(x), a_2(x), \cdots$ is bounded, has measure zero.*

Let the set $E[a_i(x) \leqq k; i=1, 2, \cdots, m]$ be denoted by G_k^m. In the corollary to Theorem 7 we set $n_j=j$, $c_j=1$, $d_j=k$ and obtain

$$|G_k^m| = \prod_{j=1}^{m} \{1 - f(k+1)\} = \{1 - f(k+1)\}^m.$$

If we let $G_k = E[a_i(x) \leqq k; i=1, 2, \cdots]$, then $G_k \in G_k^m$, $m=1, 2, \cdots$, and so $|G_k| = 0$. The set of x, $0<x<1$, for which the sequence $a_1(x), a_2(x), \cdots$ is bounded is $G=\sum_{i=1}^{\infty} G_i$ and consequently $|G| = 0$.

Similarly the set of x, $0<x<1$, for which $a_i(x)>k$, $i=1, 2, \cdots, m$, has measure $\{f(k+1)\}^m$. An argument similar to that used in the proof of Theorem 8 proves the following theorem.

THEOREM 9. *If $f \in F_p$, then the set of x, $0<x<1$, for which $a_i(x)>1$, $i=1, 2, \cdots$, has measure zero.*

THEOREM 10. *If $f \in F_p$ and $\phi(1), \phi(2), \cdots$ is a sequence of positive integers for which*

$$\sum_{n=1}^{\infty} f(\phi(n) + 1) \tag{7.1}$$

is divergent, then the set of x, $0<x<1$, for which $a_n(x) \leqq \phi(n)$, $n=1, 2, \cdots$, has measure zero.

Let $H_m = E[a_i(x) \leqq \phi(i); i=1, 2, \cdots, m]$. By an argument similar to the one used in proving Theorem 8 we have

$$|H_m| = \prod_{i=1}^{m} \{1 - f(\phi(i) + 1)\}. \tag{7.2}$$

Since $0<f(\phi(i)+1)<1$ for $i=1, 2, \cdots$, the divergence of the series (7.1) is equivalent to the limit as $m \to \infty$ of the product in (7.2) being zero. If we let $H=E[a_i(x) \leqq \phi(i); i=1, 2, \cdots]$, then since $H \in H_m$ for every positive integer m, it follows that $|H| = 0$.

The last three theorems can be generalized to infinite subsequences of the sequence $a_1(x), a_2(x), \cdots$.

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CORNELL UNIVERSITY

COMMENT ON THE REAL LINE

PAUL R. HALMOS

Is it possible to topologize the additive group of the real line so that it becomes a *compact* topological group? It is the purpose of this note to establish the affirmative answer to this question. While the fact is not deep and the proof not difficult, they are presented here mainly to call attention to the almost completely unsolved problem of algebraic characterization of those abstract abelian groups which can be compact.

(1) If G is any additively written abelian group and n is any positive integer, consider the subgroups G_n' and G_n'' defined as follows. G_n' is the set of all x's in G for which $nx=0$; G_n'' is the set of all x's which have the form $x=ny$ for some y in G. These two subgroups are dual in the sense that if G is a locally compact abelian group and Γ is its character group, then the annihilator of G_n' is Γ_n'' and (consequently) the annihilator of G_n'' is Γ_n'. (The first statement implies that the annihilator of Γ_n'' is G_n'; the second statement follows from this by interchanging the roles of G and Γ.) To prove this let $\xi=\xi(x)$ be any element of Γ, and consider the identity $\xi(nx)=n\xi(x)$. Since $\xi(nx)$ vanishes identically if and only if ξ belongs to the annihilator of G_n'', and $n\xi(x)$ vanishes identically if and only if $\xi\in\Gamma_n'$, the asserted duality follows. It follows also that a locally compact abelian group G is divisible if and only if its character group Γ is torsion free, and therefore that G has both properties if and only if Γ has.

(2) In a torsion free abelian group the division of an element by an integer is unique (if at all possible). Therefore a torsion free group in which division is always possible admits the additive group of rationals as operators. Consequently such a group is a vector space over the rationals and its algebraic structure is uniquely characterized by its vector dimension, that is by the cardinal number α of elements in a Hamel basis. For the additive group of the real line α is the power of the continuum. The assertion of this note can be proved therefore by exhibiting a compact group G which is both torsion free and divisible and in which there is a Hamel basis having the power of the continuum.

(3) Let Γ be the discrete group of rationals and G its compact character group. It follows from (1) that G is divisible and torsion free. The elements $x\in G$ may be thought of as functions $x(\xi)$ from

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Γ to the unit interval modulo 1. If the elements $x_1, \cdots, x_n$ are rationally dependent then there exist also *integers* $k_1, \cdots, k_n$ such that $k_1x_1(\xi)+\cdots+k_nx_n(\xi)\equiv 0 \pmod 1$ for all $\xi\in\Gamma$ and therefore in particular $k_1x_1(1)+\cdots+k_nx_n(1)\equiv 0 \pmod 1$. Since the mapping $x\to x(1)$ is an *onto* homomorphism from G to the unit interval modulo 1, the existence in G of a Hamel basis with power smaller than that of the continuum leads to a contradiction. For then every subset of G with the power of the continuum would contain a rationally dependent finite set and consequently every subset of the unit interval modulo 1 with the power of the continuum would contain a finite set which is integrally dependent modulo 1. Since this last possibility is absurd, and since the power of G itself is that of the continuum, it follows that a Hamel basis in G has exactly the power of the continuum. The affirmative answer to the question of this note is thus completely established.

SYRACUSE UNIVERSITY

ON CERTAIN SYSTEMS WHICH ARE ALMOST GROUPS

HENRY B. MANN

We consider systems $\mathfrak{S}$ of elements $A, B, \cdots$ satisfying the following postulates.

I. To every pair of elements A, B in $\mathfrak{S}$ there exists a uniquely determined product $A \cdot B$ in $\mathfrak{S}$.

II. $(AB)C = A(BC)$.

III. There exists in $\mathfrak{S}$ *an* element E such that $EA = A$ for every A.

IV. To every A in $\mathfrak{S}$ there exists *an* A' in $\mathfrak{S}$ such that $AA' = E$.

The system $\mathfrak{S}$ differs from a group only in that it contains a left unit and right inverse instead of a right unit and right inverse. We shall call such a system a left right system, abbreviated (l, r) system.

DEFINITION 1. *An element F is called an idempotent if $F^2 = F$.*

PROPOSITION 1. *If F is idempotent then $FE = E$.*

PROOF. From $FF = F$ we have $FFF^{-1} = FF^{-1}$, hence $FE = E$.

PROPOSITION 2. *If $FE = E$ then F is idempotent.*

For $FF = FEF = EF = F$.

PROPOSITION 3. *The idempotents of a (l, r) system form a (l, r) system.*

PROOF. If F and F' are idempotent then by Proposition 1

$$FF'E = FE = E,$$

hence FF' is idempotent and postulate I is fulfilled. Postulate II is fulfilled since it is fulfilled in $\mathfrak{S}$. E itself is idempotent and is by Proposition 1 a right inverse to every idempotent. Hence all the postulates of a (l, r) system are fulfilled.

PROPOSITION 4. *If F is idempotent and A is any element then $FA = A$.*

For $FA = FEA = EA = A$.

DEFINITION 2. *A (l, r) system which contains only idempotents will be called an idempotent (l, r) system.*

From Proposition 4 we see that in every idempotent (l, r) system every product is equal to its last factor. On the other hand it is easy to verify that every system of elements in which every product is

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equal to its last factor is an idempotent (l, r) system since every element of such a system can be considered the unit element and is then inverse to all elements. From the foregoing we have the following theorem.

THEOREM 1. *There exists one and only one idempotent (l, r) system of order n ($n=1, 2, \cdots$ ad inf.). It consists of the elements $A_1, A_2, \cdots, A_n$ with the law of composition $A_iA_j=A_j$.*

Remark. Similarly one and only one idempotent (l, r) system exists for every cardinal number.

PROPOSITION 5. *The elements of any (l, r) system which are of the form AF where F is a fixed idempotent form a group.*

PROOF. $(AF)(BF)=A(FB)F=ABF$ by Proposition 4. The composition is by hypothesis associative, F is a right unit and $A^{-1}F$ a right inverse. Since the system forms a group we must have $A^{-1}FAF=F$.

Let $\mathfrak{J}=(E, F_2, \cdots)$ be an idempotent (l, r) system, $\mathfrak{G}=(E, A_2, \cdots)$ a group. We form a system $\mathfrak{S}$ in the following way: $\mathfrak{S}$ consists of all pairs A_iF_j with the following rule of combination:

$$A_iF_jA_kF_l = A_iA_kF_l.$$

It is easy to verify that $\mathfrak{S}$ is a (l, r) system.

DEFINITION 3. *The system $\mathfrak{S}$ defined in the preceding paragraph is called the direct product of $\mathfrak{J}$ and $\mathfrak{G}$.*

PROPOSITION 6. *Every element of a (l, r) system may be written in the form $(AE)F$ where F is idempotent.*

PROOF. We have $(A^{-1}A)(A^{-1}A)=A^{-1}(AA^{-1})A=A^{-1}(EA)=A^{-1}A$. Hence $A^{-1}A$ is idempotent. But $A=AA^{-1}A=A(EA^{-1}A)=AE(A^{-1}A)$.

PROPOSITION 7. *If $AEF=BEF'$ where F and F' are idempotents then $AE=BE$ and $F=F'$.*

PROOF. From $AEF=BEF'$ we have $AEFE=BEF'E$. Hence on account of Proposition 1, $AE=BE$. Since the elements of the form AE form a group, $A^{-1}E$ is a left inverse of AE and $F=F'$ follows.

From Propositions 5, 6 and 7 we have the following theorem.

THEOREM 2. *Every (l, r) system is the direct product of an idempotent (l, r) system and a group.*

PROPOSITION 8. *Every idempotent F of a (l, r) system $\mathfrak{S}$ is a unit element of $\mathfrak{S}$.*

For by Proposition 4, $FA = A$ for every A and $A(A^{-1}F) = EF = F$.

Let $S_1, \cdots, S_n$ be n statements. Let A_i be the statement: "All the preceding statements are annulled but S_i is true." It is interesting to note that the statements A_i form an idempotent (l, r) system.

OHIO STATE UNIVERSITY

A CONJECTURE IN ELEMENTARY NUMBER THEORY

LEON ALAOGLU AND PAUL ERDŐS

A well known conjecture of Catalan states that *if $f(n)$ is the sum of all divisors of n except n, then the sequence of iterates of $f(n)$ is either eventually periodic or ends at* 1. It not only seems impossible to prove this, but it is also very difficult to verify.[1]

Another conjecture of Poûlet,[2] which appears equally difficult to prove, has the doubtful merit that it is easy to verify. Let $\sigma(n)$ be the sum of all divisors of n, and let $\phi(n)$ be Euler's function. Then *for any integer n the sequence*

$$f_0(n) = n, \qquad f_{2k+1}(n) = \sigma(f_{2k}(n)), \qquad f_{2k}(n) = \phi(f_{2k-1}(n))$$

is eventually periodic.

We have verified this conjecture to $n = 10000$ (extending Poulet's verification) by using Glaisher's tables.[3] The checking was facilitated by the following observation: if the conjecture is to be checked for all $n < x$, it is enough to find a member of the sequence other than the first which is less than x.

The longest cycle found was in the sequence $f_k(9216)$. It starts with $f_8(9216)$, and is: 34560, 122640, 27648, 81800, 30976, 67963, 54432, 183456, 48384, 163520, 55296, 163800, 34560. However our method of checking does not show that this is the largest cycle up to 10000, and in fact Poulet found that $f_k(1800)$ has the same length 12.

As a rule $\phi(\sigma(n))$ is less than n. In fact, it can be shown that *for every $\epsilon > 0$, $\phi(\sigma(n)) < \epsilon n$, except for a set of density* 0. The proof follows from the following two observations:

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[2] P. Poulet, *Nouvelles suites arithmétiques*, Sphinx vol. 2 (1932) pp. 53–54.

[3] J. W. L. Glaisher, *Number-divisor tables*, British Association for the Advancement of Science, Mathematical Tables, vol. 8.

(1) *For a given prime p, the set of all n such that $\sigma(n) \equiv 0 \pmod{p}$ is of density* 1.

The set of all integers not divisible by any prime q of the form $px-1$ is of density zero, since $\sum_q 1/q$ diverges. Hence the set of all integers divisible by a prime $q>N$ of this type is of density 1. But the set of all integers divisible by q^2, $q>N$, is of density less than $\sum_{q>N} 1/q^2 = o(1)$. Therefore, if x is large, the number of n less than x such that $\sigma(n) \equiv 0 \pmod{p}$ exceeds $(1-\epsilon)x$.

(2) *Except for ϵx integers n less than x, $\sigma(n) < c(\epsilon)n$.*

This follows from the fact that $\sum_{n<x} \sigma(n) \sim \pi^2 n^2/12$.

Choose p so that $\prod_{q \leq p} (1-1/q) < \delta/c(\epsilon)$. Then, if x is sufficiently large, all except $\eta x + \epsilon x$ integers n less than x have $\sigma(n) < c(\epsilon)n$, $\sigma(n) \equiv 0 \pmod{q}$ for all $q \leqq p$. But, with these exceptions, $\phi[\sigma(n)] < \delta n$, which completes the proof, since η and ϵ are arbitrary.

In much the same way it can be shown that *for every $c>0$, $\sigma[\phi(n)] > cn$ except for a set of density zero.*

Actually, much more can be shown. *Except for a set of density zero,* $e^{\gamma}\phi[\sigma(n)] \log\log\log n \sim \sigma(n)$, *and* $e^{-\gamma}\sigma[\phi(n)]/\log\log\log n \sim \phi(n)$, where γ is Euler's constant. The proof is suppressed, but it might be noted that the reason for this result is that, for almost all n, $\phi(n)$ and $\sigma(n)$ are both divisible by all primes less than $(\log\log n)^{1-\epsilon}$, and by relatively few primes greater than $(\log\log n)^{1+\epsilon}$.

There exist numbers for which $\phi(\sigma(n)) = n$. Up to 2500 these numbers are 1, 2, 8, 12, 128, 240, 720; while two further solutions are 2^{15} and 2^{31}. Poulet gives many others; we do not know whether there are infinitely many solutions.

We state two further conjectures:

(a) Form the sequence $\sigma(n)$, $\sigma(\sigma(n))$, $\phi(\sigma(\sigma(n)))$, $\sigma(\phi(\sigma(\sigma(n))))$ in which the functions are successively applied in the order $\sigma, \sigma, \phi, \sigma, \sigma, \phi, \sigma, \sigma, \phi, \cdots$. This sequence seems to tend to infinity if n is large enough.

(b) On the other hand, the sequence $\phi(n), \phi(\phi(n)), \sigma(\phi(\phi(n))), \cdots$, in which the order is $\phi, \phi, \sigma, \phi, \phi, \sigma, \phi, \phi, \sigma, \cdots$, seems to converge to 1, for all n.

Obviously many more such conjectures can be formulated.

PURDUE UNIVERSITY

CONGRUENCES INVOLVING THE PARTITION FUNCTION $p(n)$

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1. Introduction. The purpose of this note is to give certain congruence properties of $p(n)$, the partition function, for moduli 13 and 17, analogous to those obtained by Ramanujan for moduli 5, 7 and 11. The method and notation employed are essentially those of Ramanujan in [1].[1] Let

$$P = P(x) = 1 - 24\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n},$$

$$Q = Q(x) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3x^n}{1-x^n},$$

$$R = R(x) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5x^n}{1-x^n},$$

$$f(x) = \prod_{i=1}^{\infty}(1-x^i).$$

Then it is known that

$$f(z) = 1 - x - x^2 + x^5 + x^7 - \cdots$$

$$= \sum_{n=-\infty}^{+\infty} (-1)^n x^{n(3n-1)/2}, \tag{1}$$

$$Q^3 - R^2 = 1728x[f(x)]^{24}. \tag{2}$$

Furthermore, let $\Phi_{r,s}(x) = \sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n$, where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n. In particular

$$\Phi_{0,s}(x) = \sum_{n=1}^{\infty} \frac{n^s x^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma_s(n)x^n,$$

so that

$$P = 1 - 24\Phi_{0,1}(x),$$

$$Q = 1 + 240\Phi_{0,3}(x),$$

$$R = 1 - 504\Phi_{0,5}(x).$$

Then in terms of the functions $\sum_{r,s}(n)$, defined by

$$\sum_{r,s}(n) = \sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m),$$

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

the main result for modulus 13 is given by the congruence (10) of §2,

$$(10) \qquad p(13n-7)-p(13n-176)-\cdots \equiv -2\Sigma_{5,5}(13n) \pmod{13},$$

and for modulus 17 by the congruence (14) of §3,

$$(14) \qquad p(17n-12)-p(17n-301)-\cdots \equiv \Sigma_{5,9}(17n) \pmod{17}.$$

Another interesting congruence for modulus 13 is found in (12) of §2, namely

$$(12) \qquad p(13n-7) \equiv -2\sum_{m=1}^{n} \tau(m)p\left(\frac{n-m}{13}\right) \pmod{13},$$

where $\tau(m)$ is the Ramanujan function defined by

$$(3) \qquad \sum_{m=1}^{\infty} \tau(m)x^m = x[f(x)]^{24},$$

that is, $\tau(m)$ is the coefficient of x^m in the expansion of $x[f(x)]^{24}$.

2. **Congruence properties, modulus 13.** We let I denote a power series in x with integral coefficients. Now $\Phi_{r,s}(x)=\vartheta^r\Phi_{0,s-r}(x)$, where $\vartheta=xd/dx$. Also $\vartheta P=-24\Phi_{1,2}(x)$, $\vartheta Q=240\Phi_{1,4}(x)$, and $\vartheta R=-504\Phi_{1,6}(x)$. Then from Tables I, II and III of [2] we may form the following tables for expressing $\Phi_{r,s}(x)$ in terms of P, Q, and R. Each entry has been reduced modulo 13 so that to each must be added $13I$. Use has also been made of the relations

$$(4) \qquad Q^3 = 3R^2 - 2 + 13I,$$

$$(5) \qquad Q^2R = P + 13I.$$

TABLE I

$$\Phi_{0,1}(x) = -6P + 6$$
$$\Phi_{0,3}(x) = -2Q + 2$$
$$\Phi_{0,5}(x) = -4R + 4$$
$$\Phi_{0,7}(x) = -Q^2 + 1$$
$$\Phi_{0,9}(x) = 3QR - 3$$
$$\Phi_{1,2}(x) = 6P^2 - 6Q$$
$$\Phi_{1,4}(x) = 5R - 5PQ$$
$$\Phi_{1,6}(x) = -2PR + 2Q^2$$
$$\Phi_{1,8}(x) = -5PQ^2 + 5QR$$
$$\Phi_{1,10}(x) = -4PQR + R^2 + 3$$

TABLE II

$$\Phi_{2,3}(x) = P^3 - 3PQ + 2R$$
$$\Phi_{2,5}(x) = -P^2Q + 2PR - Q^2$$
$$\Phi_{2,7}(x) = P^2R - 2PQ^2 + QR$$
$$\Phi_{2,9}(x) = 6P^3Q^2 + PQR + 4R^3 + 2$$
$$\Phi_{2,11}(x) = 5P^2QR + 4P + 4PR^2$$
$$\Phi_{3,4}(x) = -3P^4 + 5P^2Q + 2PR - 4Q^2$$
$$\Phi_{3,6}(x) = 6P^3Q - 5P^2R + 5PQ^2 - 6QR$$
$$\Phi_{3,8}(x) = 5P^3R - 2P^2Q^2 + 2PQR + 3R^2 + 5$$
$$\Phi_{3,10}(x) = 5P^3Q^2 - 2P^2QR - 3PR^2$$
$$\Phi_{3,12}(x) = 5P^3QR - 6P^3 + 6P^2R^2 + 4QR^2 + 4Q$$
$$\Phi_{4,5}(x) = -P^5 - 3P^3Q + 6P^2R + 2PQ^2 - 4QR$$
$$\Phi_{4,7}(x) = -3P^4Q - P^3R - 5P^2Q^2 - PQR - 3$$
$$\Phi_{4,9}(x) = -6P^4R - 2P^3Q^2 + 3P^2QR - 4PR^2 - 4P$$
$$\Phi_{4,11}(x) = -3P^4Q^2 - P^3QR + P^2R^2 + 6P^2 - 3QR^2$$
$$\Phi_{5,6}(x) = 5P^6 + 3P^4Q + 5P^3R - 4P^2Q^2 + 3PQR - 4R^3 + 5$$

Now let $\Delta = (Q^3 - R^2)$. Then by the use of (4) and (5) we may express Δ^7 in terms of P, Q, and R as follows:

$$\Delta^7 = -5P^6 - 2P^4Q + 6P^3R - 6P^2Q^2 - 6PQR - 2R^2 + 2 + 13I.$$

Then, making use of Tables I and II,

$$\Delta^7 = -\Phi_{5,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{3,8}(x) - 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) - 3\Phi_{0,5}(x) + 2[\Phi_{0,5}(x)]^2 + 13I.$$

But since

$$\Phi_{0,5}(x) = \sum_{n=1}^{\infty} \sigma_5(n)x^n,$$

we have

$$[\Phi_{0,5}(x)]^2 = \left[\sum_{n=1}^{\infty} \sigma_5(n)x^n\right]\left[\sum_{m=1}^{\infty} \sigma_5(m)x^m\right] = \sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m)x^n,$$

and then

$$\begin{aligned}\Delta^7 = &- \Phi_{6,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{2,8}(x) \\ &- 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) - 3\Phi_{0,8}(x) \\ &+ 2\sum_{n=2}^{\infty} (\Sigma'_{5,5}(n))x^n + 13I,\end{aligned} \tag{6}$$

where[2]

$$\Sigma'_{5,5}(n) = \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).$$

Instead of equation (6) we may obtain a slightly different form for Δ^7 by introducing Ramanujan's function $\tau(n)$, defined by equation (3). Then

$$Q^3 - R^2 = (3R^2 - 2 - R^2) + 13I = 2(R^2 - 1) + 13I$$

and

$$\begin{aligned}6(R^2 - 1) &= -3x[f(x)]^{24} + 13I \\ &= -3\sum_{n=1}^{\infty} \tau(n)x^n + 13I,\end{aligned}$$

so that

$$\begin{aligned}\Delta^7 = &- \Phi_{6,6}(x) + 4\Phi_{4,7}(x) + 3\Phi_{2,8}(x) \\ &- 6\Phi_{2,9}(x) + 3\Phi_{1,10}(x) \\ &- 3\sum_{n=1}^{\infty} \tau(n)x^n + 13I.\end{aligned} \tag{7}$$

Now, by (2),

$$\begin{aligned}\Delta^7 = (Q^3 - R^2)^7 &= \{1728x[f(x)]^{24}\}^7 \\ &= -x^7 f(x^{169})/f(x) + 13I.\end{aligned}$$

But by (1),

$$\begin{aligned}\frac{x^7 f(x^{169})}{f(x)} &= x^7 \left\{\sum_{n=-\infty}^{+\infty} (-1)^n x^{169n(3n-1)/2}\right\}\left\{\sum_{m=0}^{\infty} p(m)x^m\right\} \\ &= \sum_{n=7}^{\infty} \{p(n-7) - p(n-176) - p(n-345) + \cdots\}x^n,\end{aligned}$$

where the numbers 7, 176, 345, $\cdots$, are alternately of the form $(13m-2)(39m-7)/2$ and $(13m+2)(39m+7)/2$. We therefore have

[2] Ramanujan defines $\Sigma_{r,s}(n) = \sum_{m=0}^{n} \sigma_r(m)\sigma_s(n-m)$ so that $\Sigma_{5,5}(n) = \Sigma'_{5,5}(n) + 2\sigma_5(0)\sigma_5(n)$, $\sigma_5(0) = \zeta(-5)/2 = -1/504$.

$$(8)\qquad \Delta^7 = -\sum_{n=7}^{\infty}\{p(n-7)-p(n-176)-p(n-345)+\cdots\}x^n+13I.$$

Comparing coefficients of x^n in (6) and (8) we obtain

$$(9)\qquad \begin{aligned} p(n-7)-p(n-176)-p(n-345)+\cdots \\ \equiv +n^5\sigma_1(n)-4n^4\sigma_3(n)-3n^3\sigma_5(n) \\ +6n^2\sigma_7(n)-3n\sigma_9(n)+3\sigma_5(n) \\ -2\Sigma'_{5,5}(n) \pmod{13}. \end{aligned}$$

If we replace n by $13n$ we get

$$(10)\qquad \begin{aligned} p(13n-7)-p(13n-176)-\cdots \\ \equiv 3\sigma_5(13n)-2\Sigma'_{5,5}(13n) \pmod{13} \\ \equiv -2\Sigma_{5,5}(13n) \pmod{13}. \end{aligned}$$

In a similar fashion, from (7) and (8) we obtain

$$(11)\qquad p(13n-7)-p(13n-176)-\cdots \equiv 3\tau(13n) \pmod{13}.$$

We next wish to compare this result with that obtained by Zuckerman [3]. For this we rewrite (11) as follows

$$\sum_{n=1}^{\infty}p(13n-7)x^{13n}\sum_{n=-\infty}^{+\infty}(-1)^n x^{169n(3n+1)/2}=3\sum_{n=1}^{\infty}\tau(13n)x^{13n}+13I.$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty}p(13n-7)x^{13n} &= 3\sum_{n=1}^{\infty}\tau(13n)x^{13n}\frac{1}{f(x^{169})}+13I \\ &= 3\sum_{n=1}^{\infty}\tau(13n)x^{13n}\sum_{n=0}^{\infty}p\left(\frac{n}{13}\right)x^{13n}+13I, \end{aligned}$$

$$p(13n-7)\equiv 3\sum_{m=1}^{n}\tau(13m)p\left(\frac{n-m}{13}\right) \pmod{13}.$$

But since

$$\tau(sp^{\lambda})=\tau(p)\tau(sp^{\lambda-1})-p^{11}\tau(sp^{\lambda-2}),\qquad (s,p)=1,$$

we have

$$\begin{aligned} \tau(13m) &= \tau(13^a m'),\ (m',13)=1, \\ &\equiv \tau(13)\tau(13^{a-1}m') \pmod{13} \\ &\equiv -5\tau(m) \pmod{13}, \end{aligned}$$

and so

$$(12) \qquad p(13n-7) \equiv -2\sum_{m=1}^{n} \tau(m)p\left(\frac{n-m}{13}\right) \pmod{13}.$$

Zuckerman's result gives

$$\sum_{n=0}^{\infty} p(13n+6)x^n = -2\frac{f(x^{13})}{f(x)} + 13I$$

from which equation (11) may also be obtained.

3. **Congruence properties, modulus 17.** Let I again denote a power series in x with integral coefficients. As before we express each $\Phi_{r,n}(x)$ in terms of P, Q and R. To each entry must be added $17I$.

TABLE III

$\Phi_{0,1}(x) = -5P + 5$
$\Phi_{0,3}(x) = -8Q + 8$
$\Phi_{0,5}(x) = 3R - 3$
$\Phi_{0,7}(x) = -4Q^2 + 4$
$\Phi_{0,9}(x) = -2QR + 2$
$\Phi_{0,11}(x) = 8Q^3 + 6R^2 + 3$
$\Phi_{0,13}(x) = -5Q^2R + 5$

$\Phi_{1,2}(x) = P^2 - Q$
$\Phi_{1,4}(x) = 3PQ - 3R$
$\Phi_{1,6}(x) = -7PR + 7Q^2$
$\Phi_{1,8}(x) = 3PQ^2 - 3QR$
$\Phi_{1,10}(x) = 4PQR + Q^3 - 5R^2$
$\Phi_{1,12}(x) = 8PQ^3 + 6PR^2 + 3Q^2R$
$\Phi_{1,14}(x) = -3PQ^2R + 8QR^2 - 5$

$\Phi_{2,3}(x) = 3P^3 + 8PQ + 6R$
$\Phi_{2,5}(x) = -3P^2Q + 6PR - 3Q^2$
$\Phi_{2,7}(x) = 3P^2R - 6PQ^2 + 3QR$
$\Phi_{2,9}(x) = -2P^2Q^2 + 4PQR - 3Q^3 + R^2$
$\Phi_{2,11}(x) = -2P^2QR - PQ^3 + 5PR^2 - 2Q^2R$
$\Phi_{2,13}(x) = 3P^2Q^3 - 2P^2R^2 - 2PQ^2R + 8QR^2 - 7$
$\Phi_{2,15}(x) = -8P^2Q^2R + 3PQR^2 + 5P$

$$\Phi_{2,4}(x) = 5P^4 + 4P^2Q + 6PR + 2Q^2$$
$$\Phi_{2,6}(x) = 7P^3Q - 4P^3R + 4PQ^2 - 7QR$$
$$\Phi_{2,8}(x) = 2P^6R - 6P^2Q^2 + 6PQR - Q^3 - R^2$$
$$\Phi_{2,10}(x) = 4P^5Q^2 + 5P^3QR + PQ^3 - 6PR^2 - 4Q^2R$$
$$\Phi_{2,12}(x) = -2P^5QR + 7P^2Q^3 - P^3R^2 - 6PQ^2R + 7QR^2 - 5$$
$$\Phi_{2,14}(x) = -5P^3Q^3 - 8P^3R^2 + P^2Q^2R + 7P - 6PQR^2 - 6Q^3R$$
$$\Phi_{2,16}(x) = -5P^3Q^2R + 6P^2QR^2 + 8P^2 + 4PQ^3R - PR^3 + Q + 4Q^2R^2$$

$$\Phi_{4,5}(x) = -4P^5 + 6P^3Q + 5P^2R - 8PQ^2 + QR$$
$$\Phi_{4,7}(x) = -3P^4Q - 5P^3R - P^2Q^2 - 5PQR + 6Q^3 + 8R^2$$
$$\Phi_{4,9}(x) = -7P^4R - 6P^3Q^2 - 8P^2QR - 3PQ^3 - 3PR^2 - 7Q^2R$$
$$\Phi_{4,11}(x) = -2P^4Q^2 + 8P^3QR - P^2Q^3 + 6P^2R^2 + 8PQ^2R - QR^2 - 1$$

$$\Phi_{5,6}(x) = 4P^6 + 8P^4Q + 7P^3R + 7P^2Q^2 - 6PQR + 3Q^3 - 6R^2$$
$$\Phi_{5,8}(x) = -2P^5Q - 7P^4R - 3P^3Q^3 + 3P^2QR + 3PQ^3 + 4PR^2 + 2Q^2R$$
$$\Phi_{5,10}(x) = -3P^5R - 2P^4Q^2 + 4P^3QR - 2P^2Q^3 - 2P^2R^2 + 2PQ^2R + 2QR^2 + 1$$

$$\Phi_{6,7}(x) = 2P^7 - 8P^5Q + 4P^4R - 6P^2Q^2 - 2P^2QR + 2PQ^3 - 4PR^2 - 5Q^2R$$
$$\Phi_{6,9}(x) = 7P^4Q - 8P^3R + 3P^4Q^2 - 4P^3QR - 6P^2Q^3 - 8P^2R^2 - 8PQ^2R - 4QR^2 - 6$$
$$\Phi_{7,8}(x) = 4P^8 + 7P^6Q + 6P^5R - 7P^4Q^2 - 5P^3QR - P^2Q^3 + 2P^2R^2 + 5PQ^2R + 5QR^2 + 1$$

In the reduction of the powers of Q and R in the expansion of $(Q^3 - R^2)^{12}$ we have made use of the relations

$$Q^4 = 3QR^2 - 2 + 17I, \qquad R^3 = 4Q^3R - 3P + 17I,$$

together with combinations of these to form

$$Q^4R = 7PQ - 6R + 17I,$$
$$QR^3 = 8PQ - 7R + 17I,$$
$$Q^5R = 4PQ^2R^2 - 5PQ + 2R + 17I,$$
$$Q^5R^4 = 5P^2Q^2 + 5PQR + 8R^3 + 17I,$$
$$QR^6 = 6Q^3R^2 - 4P^2Q - PR + 17I.$$

We obtain

$$\Delta^{12} = (Q^3 - R^2)^{12} = -P^8 - 4P^6Q + 6P^5R + 3P^4Q^2 + 4P^3QR + 6P^2Q^3 - 3P^2R^2 + 2PQ^2R + 6QR^2 - 2 + 17I.$$

Making use of the relations in Table III we have

$$\Delta^{12} = 4\Phi_{7,8}(x) - 7\Phi_{6,9}(x) + 2\Phi_{5,10}(x) + 6\Phi_{4,11}(x) - 4\Phi_{3,12}(x) + 2\Phi_{2,13}(x) - 8\Phi_{1,14}(x) + 3P^2R^2 + 2QR^2 - 5 + 17I.$$

But

$$3P^2R^2 + 2QR^2 - 5 = 6\Phi_{2,7}(x)\Phi_{0,8}(x) + 6\Phi_{0,9}(x)\Phi_{0,8}(x) + \Phi_{2,7}(x) - 2\Phi_{1,14}(x) + \Phi_{0,9}(x) + 5\Phi_{0,8}(x) + 17I.$$

Therefore

$$\Delta^{12} = 4\Phi_{7,8}(x) - 7\Phi_{6,9}(x) + 2\Phi_{5,10}(x) + 6\Phi_{4,11}(x) - 4\Phi_{3,12}(x) + 2\Phi_{2,13}(x) + 7\Phi_{1,14}(x) + \Phi_{2,7}(x) + \Phi_{0,9}(x) + 5\Phi_{0,8}(x) + 6\Phi_{2,7}(x)\Phi_{0,8}(x) + 6\Phi_{0,9}(x)\Phi_{0,8}(x) + 17I.$$

Now we also know that

$$\Delta^{12} = -4x^{12}[f(x)]^{289}/f(x) + 17I = -4x^{12}f(x^{289})/f(x) + 17I,$$

and therefore

$$x^{12}f(x^{289})/f(x) = -\Phi_{7,8}(x) + 6\Phi_{6,9}(x) + 8\Phi_{5,10}(x) + 7\Phi_{4,11}(x) + \Phi_{3,12}(x) + 8\Phi_{2,13}(x) - 6\Phi_{1,14}(x) + 4\Phi_{2,7}(x) + 4\Phi_{0,9}(x) + 3\Phi_{0,8}(x) + 7\Phi_{2,7}(x)\Phi_{0,8}(x) + 7\Phi_{0,9}(x)\Phi_{0,8}(x) + 17I. \tag{13}$$

Equating the coefficients of x^n on both sides of (13) we obtain

$$p(n-12) - p(n-301) - \cdots \equiv -n^7\sigma_1(n) + 6n^6\sigma_2(n) + 8n^5\sigma_5(n) + 7n^4\sigma_7(n) + n^3\sigma_9(n) + 8n^2\sigma_{11}(n) - 6n\sigma_{12}(n) + 4n^2\sigma_8(n) + 4\sigma_9(n) + 3\sigma_8(n) + 7\sum_{m=1}^{n-1} m^2\sigma_5(m)\sigma_8(n-m) + 7\sum_{m=1}^{n-1}\sigma_8(m)\sigma_9(n-m) \pmod{17},$$

where the numbers 12, 301, $\cdots$ are alternately of the form

$(17n-3)(51n-8)/2$ and $(17n+3)(51n+8)/2$. Now replacing n by $17n$ we get

$$p(17n-12)-p(17n-301)-\cdots\equiv 4\sigma_9(17n)+3\sigma_5(17n)+7\sum_{m=1}^{17n-1}m^2\sigma_5(m)\sigma_5(17n-m)+7\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m)\pmod{17}.$$

In order to further simplify this result we note that

$$\Phi_{0,5}(x)\Phi_{2,7}(x)-4[\Phi_{1,6}(x)]^2=-3\Phi_{2,7}(x)+3\Phi_{0,5}(x)\Phi_{0,9}(x)-8\Phi_{0,9}(x)-6\Phi_{0,5}(x)+17I$$

and therefore

$$\sum_{m=1}^{n-1}m^2\sigma_5(m)\sigma_5(n-m)\equiv 4\sum_{m=1}^{n-1}m(n-m)\sigma_5(m)\sigma_5(n-m)-3n^2\sigma_5(n)+3\sum_{m=1}^{n-1}\sigma_5(m)\sigma_9(n-m)-8\sigma_9(n)-6\sigma_5(n)\pmod{17}.$$

But

$$\sum_{m=1}^{n-1}m(n-m)\sigma_5(m)\sigma_5(n-m)=n\sum_{m=1}^{n-1}m\sigma_5(m)\sigma_5(n-m)-\sum_{m=1}^{n-1}m^2\sigma_5(m)\sigma_5(n-m)$$

and

$$\sum_{m=1}^{n-1}m\sigma_5(m)\sigma_5(n-m)=\frac{n}{2}\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m)$$

so that finally

$$\sum_{m=1}^{17n-1}m^2\sigma_5(m)\sigma_5(17n-m)\equiv 4\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m)-5\sigma_9(17n)-8\sigma_5(17n)\pmod{17}.$$

Therefore

$$p(17n-12)-p(17n-301)-\cdots\equiv\sum_{m=1}^{17n-1}\sigma_5(m)\sigma_9(17n-m)+3\sigma_9(17n)-2\sigma_5(17n)\pmod{17},$$

$$p(17n-12)-p(17n-301)-\cdots\equiv\Sigma_{5,9}(17n)\pmod{17}. \tag{14}$$

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AUTOMORPHISMS OF FIELDS OF FORMAL POWER SERIES

O. F. G. SCHILLING

We propose to discuss in this note on power series fields in one variable the special automorphisms which do not alter the fields of coefficients. It will be proved that the pseudo-ramification groups introduced by MacLane are universal ramification groups, in the sense that a special ramification group must always be a subgroup of a well determined pseudo-ramification group. Finally we interpret the automorphism group of the field as an automorphism group of an infinite Lie ring.

Let Ω be an arbitrary field of characteristic χ. In the sequel we shall consider the field F of all formal power series $a=\sum_{j>-\infty}^{\infty}\omega_j t^j$ where the ω_j are in Ω and t is a transcendental element over Ω.[1] The field F is complete with respect to the rank one valuation V defined by $Va=m$ where m is the smallest subscript j for which $\omega_j\neq 0$. Let $\mathfrak{O}$ be the valuation ring of all holomorphic series and $\mathfrak{P}=(t)$ the principal prime ideal of $\mathfrak{O}$.

Suppose that S is an automorphism of F. We show that $\mathfrak{O}^S$ is also a valuation ring of F. For the proof[2] let a, b be any two nonzero elements of F. We must show that at least one of the quotients a/b, b/a lies in $\mathfrak{O}^S$. By assumption on S there exist unique elements c, d with $c^S=a$, $d^S=b$. Now observe that at least one of the quotients c/d or d/c lies in $\mathfrak{O}$ for $\mathfrak{O}$ is a valuation ring. Therefore at least one of the

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[1] For the basic properties of valuations see [1, 4, 5, 10]. Numbers in brackets refer to the bibliography at the end of the paper.

[2] See [4, p. 165].

elements $(c/d)^S = a/b$ or $(d/c)^S = b/a$ lies in $\mathfrak{O}^S$. Furthermore we observe that the ideal theory of $\mathfrak{O}$ is carried over isomorphically to $\mathfrak{O}^S$. Thus $\mathfrak{O}^S$ defines a rank one valuation V^S on F.

DEFINITION 1.[3] *An automorphism S of F is called analytic if $Va < Vb$ is equivalent to $V^S a^S \leqq V^S b^S$ for each pair of elements a, b in F.*

LEMMA 1. *Each automorphism of F is analytic.*

PROOF. We shall prove that F is complete with respect to the valuation V^S. Let $a_n \equiv a_{n-1} \pmod{(\mathfrak{P}^S)^n}$ be an infinite consistent system of congruences with respect to V^S. There exist uniquely determined elements c_n with $c_n^S = a_n$ and we have $c_n \equiv c_{n-1} \pmod{\mathfrak{P}^n}$. Hence there exists an element x in F with $x \equiv c_n \pmod{\mathfrak{P}^n}$, for F is complete with respect to V. Applying S to x we have $x^S \equiv c_n^S \equiv a_n \pmod{(\mathfrak{P}^S)^n}$. Hence F is complete with respect to V^S. Consequently F would be multiply complete if V and V^S were distinct valuations. Thus it would follow[4] that F is algebraically closed, in contradiction to the construction of F. Hence V and V^S are equivalent valuations, that is, S is analytic.

Suppose that s is an automorphism of Ω. If we observe the rules for the addition and multiplication of elements in F, the correspondence $\sum_{i>-\infty}^{\infty} \omega_i t^i \to \sum_{i>-\infty}^{\infty} \omega_i^s t^i = (\sum_{i>-\infty}^{\infty} \omega_i t^i)^S$ defines an automorphism of F. These automorphisms of F determine a subgroup of the automorphism group A of F which is isomorphic with the automorphism group of Ω. A simple computation shows that this subgroup is not normal if and only if it is not the trivial group consisting of the identity. Now let T be an arbitrary automorphism of F. The element T either induces an automorphism on $\Omega \subset F$ or it maps Ω into an isomorphic subfield $\Omega^T \subset F$. *We shall consider only those automorphisms of F for which all the elements of Ω are invariant.* These automorphisms form a subgroup G of A. This group G corresponds to the inertial group considered in the theory of normal algebraic extensions of fields with valuations.

LEMMA 2. *A field F which is complete with respect to a rank one valuation has no immediate extensions.*[5]

PROOF. Suppose that K is an immediate extension of F. Let A be an element of K. We shall show that A must lie in F. By assumption the value of A is the value of an element a in F. Thus A/a is a unit

[3] See [5, footnote on p. 373; 12].
[4] See [12].
[5] See [4, p. 191].

and therefore $A/a \equiv \omega_0 \pmod{\mathfrak{P}_K}$, where $\mathfrak{P}_K$ is the prime ideal of K and ω_0 is in Ω. Next there is at least one element a_1 in F with $A/a - \omega_0 \equiv a_1 \pmod{\mathfrak{P}_K^{n_1}}$, where $n_1 = V(A/a - \omega_0)$. By complete induction there exists a sequence of elements $a\omega_0$, $a\omega_0 + a a_1 \omega_1$, $\cdots$, $a\omega_0 + \cdots + a a_1 \cdots a_\nu \omega_\nu$, $\cdots$ such that $A \equiv a\omega_0 + \cdots + a a_1 \cdots a_\nu \omega_\nu \pmod{\mathfrak{P}_K^{m}}$ with $m = n_1 + \cdots + n_\nu \to \infty$ for $\nu \to \infty$. This sequence has a limit b in F and $A - b$ has value ∞ in K. Hence $A = b$ in F.[6]

Suppose that $S \neq 1$ is an automorphism of F with $\omega = \omega^S$ for all ω in Ω. Then $t^S \neq t$, for $t = t^S$ implies $a = a^S$ for every a in F. Hence each $S \neq 1$ determines by $t^S = tu(S)$ an element $u(S)$ of the unit group U of F.

THEOREM 1. *The groups G and U are in one-to-one correspondence as sets.*

PROOF. As seen above each automorphism S determines relative to the prime element t a unit $u(S)$. For the converse we shall show that the mapping $t \to t^* = tu$ determines, for given u in U, an automorphism of F. We associate to an arbitrary element $a = \sum_{\nu > -\infty}^{\infty} \omega_\nu t^\nu$ the quantity $a^* = \sum_{\nu > -\infty}^{\infty} \omega_\nu (t^*)^\nu$. The elements a^* form by definition a subfield F^* of F which is isomorphic to F. By construction the valuation V of F induces a valuation V^* on the complete field F^* with $V^* t^* = V t^* = V t$ so that F and F^* have the same residue class field. Therefore F is an immediate extension of F^* and hence $F = F^*$ by Lemma 2. Consequently the mapping $a \to a^*$ is an automorphism of F. We remark that the identity elements of G and U correspond to each other.

We next give a definition and a set of formulas which can be used to compute explicitly the coefficients of a^* relative to the prime element t. We define the derivative $D_t a$ of $a = \sum \omega_\nu t^\nu$ as $\sum \nu \omega_\nu t^{\nu-1}$ and $D_t^i = (1/i!)(d/dt)^i$. Then[7]

$$D_t^i(a + b) = D_t^i a + D_t^i b, \quad D_t^i t^n = C_{n,i} t^{n-i}, \qquad i \geqq 0,$$

$$D_t^i(\omega a) = \omega D_t^i a, \quad \omega \text{ in } \Omega,$$

$$D_t^i(ab) = \sum_{i} D_t^{i_1} a D_t^{i_2} b; \quad i_1 \geqq 0,\ i_2 \geqq 0,\ i_1 + i_2 = i.$$

As in the calculus the inversion formula of Lagrange holds, $t = \sum_{j=1}^{\infty} \lambda_j (t^*)^j$ with $\lambda_j = [D_t^j \{ D_t t^* \cdot (t/t^*)^j \}]_{t=0}$, $\lambda_1 \neq 0$. Hence t lies in F^* and thus $F^* = F$. Using the Taylor developments of the elements $a^* = \sum \omega_i (t^*)^i = \sum D_t^i \{a^*\}_{t=0} t^i$ it follows that each unit u determines by $t \to t^* = tu$ an automorphism of F. Letting $b^* = \sum \rho_i (t^*)^i = \sum D_t^i (b^*)_{t=0} t^i$ we have, by the rules on the derivatives D_t^i,

[6] For another proof see [5, pp. 379–380].

[7] See [2].

$$(a+b)^* = \sum \{D_t^j(a^*+b^*)\}_{t=0} t^j$$
$$= \sum \{D_t^j a^*\}_{t=0} t^j + \sum \{D_t^j b^*\}_{t=0} t^j = a^* + b^*,$$
$$(ab)^* = \sum \{D_t^j(a^* b^*)\}_{t=0} t^j$$
$$= \sum_j \sum \{D_t^{j_1} a^*\}_{t=0} \{D_t^{j_2} b^*\}_{t=0} t^{j_1+j_2} = a^* b^*,$$

where $j_1 \geqq 0$, $j_2 \geqq 0$, $j_1+j_2=j$, and similarly $(a\omega)^* = a^*\omega$.

Now let S, T be two automorphisms of G. On applying ST to the prime element t we find $t^{ST} = (t^T)^S = (tu(T))^S = t^S u(T)^S = tu(S)u(T)^S = tu(ST)$ where the u's are the units corresponding to the automorphisms. Thus the mapping $S \to u(S)$ gives rise to a crossed representation[8] of the automorphism group G in the unit group U, for $u(ST) = u(S)u(T)^S$. The latter relation may be viewed as a new multiplication $\times$ between the elements of U. We define $u_1 \times u_2$ to be $u_1 u_2^W$ where $W = S(u_1)$ is the automorphism which corresponds to u_1 by virtue of Theorem 1. The associativity of the group G implies that U is a group with respect to the operation $\times$.

We now proceed to a different interpretation of this operation. Let Ω_x be the set of all formal power series $\sum_{i=0}^{\infty} \omega_i x^i$, where x is an indeterminate over Ω.

DEFINITION 2. *If* $A = \sum \alpha_i x^i$ *and* $B = \beta_i x^i$ *then define* $A \otimes B$ *as* $\sum_i \beta_i (\sum_j \alpha_j x^j)^i$ *if and only if* $\alpha_0 = 0$.

From the definition of Ω_x we observe that Ω_x may be considered as an infinite vector space over Ω with restricted multiplication.[9] The element x plays the role of a left and right unit on the subset U of all elements without constant terms. The set U is a group, as may be verified by using Taylor expansions with respect to x. Obviously the product of elements corresponds to the operation of taking the function of a function. It is now easy to set up a relation between U with $\times$ as operation and U. If $u = \sum_{i=0}^{\infty} \omega_i t^i$, $\omega_0 \neq 0$, is given then we take for the corresponding element $u = \sum_{i=0}^{\infty} \omega_i x^{i+1}$. The product $u_1 \times u_2$ of two elements u_1, u_2 in U is then determined as $\{(u_1 \otimes u_2) x^{-1}\}_{x=1}$ where the factor x^{-1} is to indicate a scalar division of $u_1 \otimes u_2$ by x. In the sequel it will be useful to use the representation of the elements S in G by means of the elements in U; $S \rightleftarrows u(S) \rightleftarrows u(S)$. We shall use the symbol S ambiguously for the automorphism S and the representation $u(S)$.

Suppose now that μ_1, μ_2 are two nonzero elements of Ω. These elements determine, by $t \to t\mu_i$, automorphisms $S(\mu_i)$ in G, $i = 1, 2$. The

[8] See [7, p. 313].
[9] See [8, 9].

multiplication rule of U shows $S(\mu_1)S(\mu_2)=(x\mu_1\otimes x\mu_2)=S(\mu_1\mu_2)$ $=S(\mu_2\mu_1)=(x\mu_2\otimes x\mu_1)=S(\mu_2)S(\mu_1)$. Hence the multiplicative group Ω^* of Ω has an isomorphic image in G.

The group G contains an infinite series of subsets G_i defined by the elements $S_i=x+\sum_{\nu=i+1}^{\infty}\omega_\nu x^\nu$.

THEOREM 2. *The sets G_i are invariant subgroups of G, their intersection $\bigcap_i G_i$ is the identity and each factor group G_i/G_{i+1} is isomorphic to the additive group Ω. The factor group G/G_1 is isomorphic to the multiplicative group Ω^*.*

PROOF. Let $S=x+\omega_1x^i+\cdots$, $T=x+\omega_2x^i+\cdots$ be two elements of G_i. Then, by definition of the product in U,

$$S\otimes T = x+(\omega_1+\omega_2)x^i+\cdots,$$

$$(*)\qquad T\otimes S = x+(\omega_2+\omega_1)x^i+\cdots,$$

$$S^{-1} = x+(-\omega_1)x^i+\cdots.$$

Thus G_i is a group. To show that G_i is an invariant subgroup of G let $W=\alpha_0x+\alpha_1x^2+\cdots$ be an arbitrary element of G. The inversion formula for power series implies $W^{-1}=\alpha_0^{-1}x+(-\alpha_1/\alpha_0^2)x^2+\cdots$. Consequently, by direct computation, $W\otimes S\otimes W^{-1}=x+\alpha_0^{i-1}\omega_1x^i$ $+\cdots$ lies also in G_i. Definition of the G_i implies $G\supset G_1\supset\cdots$ and $\bigcap_i G_i=x$, the identity of G. The relations (*) imply that G_i/G_{i+1} is isomorphic to the additive group Ω. Now let $S(\alpha_0)$ be determined by the element $W=\alpha_0x+\alpha_1x^2+\cdots$. Then $S(\alpha_0)^{-1}\otimes W=\alpha_0(\alpha_0^{-1})x$ $+\alpha_1(\alpha_0^{-1})x^2+\cdots=x+(\alpha_1/\alpha_0)x^2+\cdots$ lies in G_1. Hence $G=\sum_{\alpha\in\Omega^*}S(\alpha)G_1$ and therefore $G/G_1\cong\Omega^*$.

COROLLARY 1. *The group G is a group extension of G_1 by Ω^* with factor set unity.*

PROOF. As seen before the elements $S(\mu)$, μ in Ω^*, form a subgroup of G. The elements $S(\mu)=\mu x$ induce automorphisms on the group $G_1=\{x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots\}$. The associated combinations are determined by

$$x\mu\otimes(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)=x\mu+x^2\omega_1\mu^2+\cdots$$
$$+x^{n+1}\omega_n\mu^{n+1}+\cdots,$$

$$(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)\otimes x\mu=x\mu+x^2\omega_1\mu+\cdots$$
$$+x^{n+1}\omega_n\mu+\cdots,$$

whence

$$x\mu\otimes(x+\omega_1x^2+\cdots+\omega_nx^{n+1}+\cdots)\otimes(x\mu)^{-1}$$
$$=x+x^2\omega_1\mu+\cdots+x^{n+1}\omega_n\mu^n+\cdots.$$

Since $\{S(\mu), \mu \in \Omega^*\}$ is a subgroup the factor set of Ω^* with respect to G is the unit factor set.

COROLLARY 2. *The group G_1 is a complete metric group; if the field Ω is finite then G_1 is compact.*

PROOF. The system of normal subgroups G_i defines a topology in G_1. The group G_1 is complete with respect to this topology for G_1 contains all elements $x+\omega_1 x^2+\cdots+\omega_n x^{n+1}+\cdots$ with arbitrary coefficients ω_n. We next define a metric in G_1 which is consistent with the system of defining neighborhoods G_i, $i=1, 2, \cdots$. Suppose $S = x + \alpha_1 x^2+\cdots+\alpha_{n-1}x^n+\alpha_n x^{n+1}+\cdots$, $T = x+\beta_1 x^2+\cdots+\beta_{n-1}x^n+\beta_n x^{n+1}+\cdots$ are two arbitrary elements of G_1. We define $\delta(S, T)$ to be e^{-n} if and only if $\alpha_i=\beta_i$ for all $i<n$ and $\alpha_n \neq \beta_n$. Then $\delta(S, T)=\delta(T, S)$, $\delta(S, T)>0$ for $S \neq T$, $\delta(S, T) \leqq \max[\delta(S, R), \delta(T, R)]$. Moreover G_n consists of all elements S in G_1 which have distance $\delta(1, S) \leqq e^{-n}$. Thus G_1 is a 0-dimensional metric group. The factor groups G_1/G_n contain nontrivial elements of finite order if and only if $\chi<\infty$ holds for the characteristic of Ω. If $\chi<\infty$ the elements of G_1/G_n have at most order χ^{n-1}. Hence $\lim_{n\to\infty} g^{\chi^n}=1$ for each element g of G_1, thus G_1 is a generalized χ-adic group.[10] In particular, G_1 is compact if Ω is finite. In this case $G_n/G_{n+1} \cong \Omega$, and thus G_1 is isomorphic to the inverse limit $\lim_{n\to\infty} G_1/G_n$ where the factor groups G_1/G_n are all finite.

DEFINITION 3. *The totality of elements S in G_1 with $a^S \equiv a \pmod{\mathfrak{P}^n}$ for each a in $\mathfrak{O}$ is called the nth pseudo-ramification group R_n of F.*[11]

THEOREM 3. *The groups G_n and R_n coincide.*

PROOF. We first remark that the first pseudo-ramification group R_1 coincides with G_1, for the general element $x+\omega_1 x^2+\cdots$ of G_1 induces the identity mapping on the residue class field Ω of F. Suppose now that $g=x+\sum_{\nu=n+1}^{\infty}\omega_\nu x^\nu$ is an arbitrary element of G_n. Let $a=\alpha_0+\alpha_1 t+\cdots+\alpha_{n-1}t^{n-1}+\alpha_n t^n+\cdots$. Then

$$a^g = \alpha_0 + \alpha_1\left(t+\sum_{\nu=n}^{\infty}\omega_\nu t^\nu\right)+\cdots+\alpha_{n-1}\left(t+\sum_{\nu=n}^{\infty}\omega_\nu t^\nu\right)^{n-1}$$
$$+\alpha_n\left(t+\sum_{\nu=n}^{\infty}\omega_\nu t^\nu\right)^n+\cdots$$

[10] A topological group G is called χ-adic, if $\lim_{n\to\infty} g^{\chi^n}=1$ for each g in G, $\chi<\infty$, in a sense of the topology of G. Examples of χ-adic groups are the additive subgroup of the χ-adic number field and multiplicative groups in χ-adic algebras.

[11] See [6, p. 438].

$$= \alpha_0 + \alpha_1 t + \alpha_1 \sum_{r=n}^{\infty} \omega_r t^r + \cdots + \alpha_{n-1} t^{n-1} + \alpha_{n-1}(n-1) t^n \omega_n$$
$$+ \cdots + \alpha_n t^n + \cdots + \alpha_n n t^{n+r} \omega_n + \cdots$$
$$\equiv \alpha_0 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1} \pmod{\mathfrak{P}^n},$$

whence $G_n \subseteq R_n$. Conversely, let $g \in R_n$ and $t^g = t + \beta_1 t^2 + \cdots + \beta_{n-1} t^n + \beta_n t^{n+1} + \cdots$. We compute

$$a^g = \alpha_0 + \alpha_1(t + \beta_1 t^2 + \cdots + \beta_{n-1} t^n + \beta_n t^{n+1} + \cdots)$$
$$+ \alpha_2(t + \beta_1 t^2 + \cdots)^2 + \cdots + \alpha_{n-1}(t + \beta_1 t^2 \cdots)^{n-1} + \cdots$$
$$\equiv \alpha_0 + \alpha_1 t + \alpha_1 \beta_1 t^2 + \cdots + \alpha_2 t^2 + 2\alpha_2 \beta_1 t^3 + \cdots + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \alpha_{n-1} t^{n-1} + \cdots$$
$$\equiv \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{n-1} t^{n-1} \pmod{\mathfrak{P}^n}.$$

Reducing this congruence successively mod $\mathfrak{P}^2, \mathfrak{P}^3, \cdots, \mathfrak{P}^{n-1}$ we obtain $\beta_1 = \cdots = \beta_{n-2} = 0$, whence $G_n \supseteq R_n$.

REMARK. As in the ordinary Hilbert theory it follows that R_n is an invariant subgroup of G_1. This furnishes a new proof for the invariance of the groups G_n.

THEOREM 4. *Suppose that H is a finite subgroup of G_1 and let K be the associated subfield of F. The ramification groups H_i of F/K are equal to $G_i \cap H$.*

PROOF. Let V_K be the valuation which is induced by V on the subfield K. Suppose that $\Gamma(F)$ and $\Gamma(K)$ are the value groups associated to V and V_K, respectively. We first observe that Ω is contained in K for $H \subset G_1$. Let $\overline{K}$ be the completion of K with respect to V_K, then $K \subseteq \overline{K} \subseteq F$ for the fundamental sequences in K are special fundamental sequences in F by the definition of V_K. Next observe that V_K has only V for its prolongation to F. Remark that all prolongations of V_K are given as V^h, h in H; thus, by Lemma 1, $V^h = V$. Combining these facts, we find $[F:K] = [\Gamma(F):\Gamma(K)] = [\Gamma(F):\Gamma(\overline{K})] = [F:\overline{K}]$. Consequently $K = \overline{K}$, that is, K is a complete field and H is the decomposition and inertial group for F/K. Now let H_i be the ith ramification group of F/K. Then $H_i \subset R_i = G_i$ by the proof of Theorem 3. Thus $H_i \subseteq R_i \cap H$. Conversely, let S be an element of $R_i \cap H$. Then $a^S \equiv a \pmod{\mathfrak{P}^i}$ by definition of R_i. Hence S lies in H_i. Therefore $H_i = R_i \cap H$ as asserted.

Let K be an arbitrary subfield of F such that $\Omega \subset K$ and $[F:K] < \infty$. The field K is then complete with respect to the valuation V_K

induced by V. For the proof we observe that the valuation V has a unique prolongation to the normal closure of F over K, for the latter is complete since it is a finite extension of F. The ramification theory implies, as at the beginning of the proof for Theorem 4, that K is complete. Now let $\pi(F)$ be any prime element of F, that is $V\pi(F)=1$, $F=\Omega\{\pi(F)\}$. Then $N_{F/K}\pi(F)$ is a prime element $\pi(K)$ for K because $V\pi(K)=[F:K]$ and $\Omega\{\pi(K)\}=K$ by Lemma 2 applied to the chain $\Omega\{\pi(K)\}\subseteq K\subset F$. We have $\pi(K)=\sum_{\nu=0}^{\infty}\omega_\nu\pi(F)^{n+\nu}$, where $[F:K]=n$ and $\omega_0\neq 0$. Applying an automorphism S of G to $\pi(F)$ we obtain a field $K^S=\Omega\{\pi(K)^S\}$ which is isomorphic to K and $K=K^S$ if and only if S is an automorphism of F/K. The group SHS^{-1} is the full automorphism group of F/K^S if H is the automorphism group of F/K. We apply these results to prove the following theorem.

THEOREM 5. *The automorphism group G of F contains elements of finite order.*

PROOF. We first observe that F always has normal completely ramified extensions M. The existence of such extensions depends on the nature of the residue class field Ω.[12] In case that n is an integer prime to the characteristic χ and Ω contains a primitive nth root of unity the field F surely has at least one cyclic completely ramified extension of degree n. For example, there is surely such a quadratic extension if $(\chi, 2)=1$ or $\chi=\infty$. If χ is finite then Ω contains some field of (χ^m-1)th roots of unity, $m\geqq 1$, and any $n\mid(\chi^m-1)$ may be used. For finite χ there are infinitely many cyclic completely ramified extensions of degree χ over F.[13] Hence we know that completely ramified normal extensions M/F exist. Since M is completely ramified, say of degree n, we have $\pi(F)=\sum_{\nu=0}^{\infty}\omega_\nu\pi(M)^{\nu+n}$, $\omega_0\neq 0$, for a prime element $\pi(M)$ of M. By the structure theory of fields of power series the fields M and F are isomorphic.[14] Suppose that ϕ is a realization of the isomorphism $M\cong F$, that is, $M^\phi=F$. We apply ϕ to the equation for $\pi(F)$ and obtain $\pi(F)^\phi=\sum_{\nu=0}^{\infty}\omega_\nu[\pi(M)^\phi]^{n+\nu}$. Then $\pi(F)^\phi$ determines by $\Omega\{\pi(F)^\phi\}$ a subfield K of F so that the Galois groups of M/F and $F/\Omega\{\pi(F)^\phi\}$ are isomorphic. Observing that the Galois group of M/F is isomorphic to a finite subgroup of G, the assertion of the theorem follows.[15]

As a special case we consider cyclic subgroups H of order n in G for which $(n, \chi)=1$. Let K be the field of invariants for H. We have

[12] See [11, 14].
[13] See [14].
[14] See [1, 6, 13, 14].
[15] See [3, p. 890].

$K=\Omega\{ut^n\}$ where u is a unit of F. Then $u=\omega v$ where $\omega\neq 0$ lies in Ω and $v\equiv 1 \pmod P$. We have $K=\Omega\{vt^n\}$, for K is an immediate extension of $\Omega\{vt^n\}$. Since $(n, \chi)=1$ there exists[16] a unit y of F with $y\equiv 1 \pmod P$, $y^n=v$. Therefore $K=\Omega\{(yt)^n\}$. Since $\Omega\{yt\}=\Omega\{t\}=F$ we obtain $H=SH_0S^{-1}$ where S is the automorphism of G_1 with $t^S=yt$ and H_0 is the finite subgroup of G generated by the element ζx, ζ a primitive nth root of unity. Finally $K=\Omega\{t^n\}$ if and only if $SH_0S^{-1}=H_0$, that is, S induces an automorphism on H_0. Expressing $K=\Omega\{t^n\}$ in terms of the prime elements we have $(yt)^n/t^n=y^n$ lies in $\Omega\{t^n\}$. Consequently y^n must be a power series $1+\sum_{j=1}^{\infty}\alpha_j(t^n)^j$. Conversely each unit y with y^n in $\Omega\{t^n\}$ gives rise to $K=\Omega\{t^n\}$. Since there are always elements y for which y^n does not lie in $\Omega\{t^n\}$ and $y^n=1+\omega t+\cdots$, $\omega\neq 0$, we have the existence of automorphisms S with $SH_0S^{-1}\neq H_0$ and $\Omega\{t^n\}\cap\Omega\{(yt)^n\}=\Omega$, as can be shown by comparing the series of the fields. This shows that the set of normal subfields K with $[F:K]<\infty$, $([F:K], \chi)=1$ is not a lattice. We remark that conditions on y may be derived to describe $\Omega\{(yt)^s\}=\Omega\{t^s\}$ for $s\mid n$.

The group G can be interpreted as a group of automorphisms of an infinite Lie ring $\mathfrak{O}_L$ over Ω. We define $\mathfrak{O}_L$ as the ring $\mathfrak{O}$, considered as an infinite vector space over Ω, in which a product $[f, g]$ is defined as follows. We set $[f, g]=(dg/dt)f-(df/dt)g$. The rules of differentiation imply that $[f, g]$ obeys the rules of the Jacobi bracket. For a basis of $\mathfrak{O}_L$ we may take the elements $e_i=t^{i+1}$, $i=-1, 0, 1, \cdots$ and $[e_i, e_j]=(j-i)e_{i+j}$. By actual computation it can be shown that the mapping

$$(**) \qquad f(t)\rightarrow f(t^S)/(dt^S/dt)$$

is an automorphism of $\mathfrak{O}_L$.[17] Moreover, distinct elements of G give rise to distinct automorphisms in the automorphism group $A(\mathfrak{O}_L)$ of $\mathfrak{O}_L$.

THEOREM 6. *The groups G and $A(\mathfrak{O}_L)$ coincide if $\chi=\infty$.*

PROOF. Suppose that Σ is an automorphism of $\mathfrak{O}_L$. Let $t^\Sigma=\Phi(t)$. We shall show that there exists an automorphism S in G so that Σ is determined by the formula $(**)$ applied to t, that is, $t^\Sigma=t^S/(dt^S/dt)$. Such an element S determines then the automorphism Σ on all of $\mathfrak{O}_L$ because the elements t^{i+1} form a basis of $\mathfrak{O}_L$. To determine S it suffices to find a unique element $\phi(t)=\alpha_1t+\alpha_2t^2+\cdots$, $\alpha_1\neq 0$, for which $t^\Sigma=\Phi(t)=f[\phi(t)]/(d\phi/dt)$. By formal integration we find

[16] See [10, p. 561; 11, p. 441].

[17] Compare [15, pp. 37–47].

$\phi(t) = \exp\left[\int \Phi^{-1}(t)dt\right]$ where $\int \Phi^{-1}(t)dt$ is the formal indefinite integral without a constant of integration. We observe that all formal operations involved can be carried out because they are determined in $\mathfrak{O}$. The automorphism S is given by $\phi(x)$ using the representation of G by U. In concluding we remark that $G \subset A(\mathfrak{O}_L)$ for $\chi < \infty$. The inequality may be explained by the fact that t^n is never the derivative of an element in F if $n \equiv 0 \pmod{\chi}$.

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THE EQUATION $x' \equiv xd - dx = b$

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Let $\mathfrak{A}$ be an associative algebra with a possibly infinite basis over a field Φ. Then if d is a fixed element in $\mathfrak{A}$, it is well known that the mapping $x \to x' \equiv [x, d] = xd - dx$ is a derivation[1] in $\mathfrak{A}$; that is,

$$(x + y)' = x' + y', \qquad (x\alpha)' = x'\alpha, \qquad (xy)' = x'y + xy'$$

for all x, y in $\mathfrak{A}$ and all α in Φ. The constants relative to such a derivation are the elements of $\mathfrak{A}$ that commute with d. We shall call an element b a *d-integral* if $b = a'$ for some element a in $\mathfrak{A}$, that is, if the equation $x' = xd - dx = b$ has a solution in $\mathfrak{A}$. Clearly if a is a solution of this equation then the totality of solutions is the set $\{a + c\}$ where c ranges over the set of d-constants. In a recent paper appearing in this Bulletin, R. E. Johnson obtained a necessary and sufficient condition that an element b be a d-integral under the assumption that $\mathfrak{A}$ is a separable algebraic division ring.[2] In this note we allow $\mathfrak{A}$ to be an arbitrary algebra but we make the assumption that d is an algebraic element in the sense that it satisfies a polynomial equation with coefficients in Φ. We obtain a necessary condition, which is equivalent to Johnson's condition when $\mathfrak{A}$ is a division ring, that b be a d-integral. If the minimum polynomial $\mu(\lambda)$ of d is relatively prime to its derivative $\mu'(\lambda)$, then it is easy to see that the condition is also sufficient and one may give an explicit formula for a solution of the equation $x' = b$. If we assume that $\mathfrak{A}$ is a simple algebra satisfying the descending chain condition for left ideals then we can show that our condition is also sufficient when $\mu(\lambda)$ is a product of distinct irreducible factors in $\Phi[\lambda]$ and in certain other cases. Here, however, we do not display a solution but merely prove its existence. Our results include, of course, Johnson's result for algebraic division rings, since the minimum polynomial of an element in such a ring is irreducible. No assumption about separability is required.

In order to obtain a condition for the solvability of the equation $x' = b$ we consider the matrices

$$(1) \qquad u = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \qquad v = \begin{pmatrix} d & b \\ 0 & d \end{pmatrix}$$

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[1] Cf. the author's paper *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 206–224.

[2] *On the equation* $x\alpha = \gamma x + \beta$ *over an algebraic division ring*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 202–208.

in the matrix algebra $\mathfrak{A}_2$ of two-rowed matrices with elements in $\mathfrak{A}$. If x is any element in $\mathfrak{A}$

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1}$$

and the equation $x'=xd-dx=b$ is equivalent to the matrix equation

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} d & b \\ 0 & d \end{pmatrix}. \tag{2}$$

Thus if b is a d-integral the matrices (1) are similar in $\mathfrak{A}_2$. We suppose now that d is an algebraic element and let

$$\phi(\lambda) = \lambda^m + \alpha_1\lambda^{m-1} + \cdots \tag{3}$$

be the minimum polynomial of d over Φ. Then it is clear that $\phi(u)=0$. Hence a necessary condition that b be a d-integral is that $\phi(v)=0$. Now

$$v^r = \begin{pmatrix} d^r & B_r \\ 0 & d^r \end{pmatrix}$$

where $B_r=\sum_{k=0}^{r-1} d^k b d^{r-k-1}$. Hence the condition that $\phi(v)=0$ is that

$$B_m + \alpha_1 B_{m-1} + \cdots + \alpha_{m-1}B_1 = 0. \tag{4}$$

As we have shown elsewhere[3]

$$B_r = C_{r,1}d^{r-1}b + C_{r,2}d^{r-2}b' + \cdots + b^{(r-1)}$$

where $b^{(k)}=(b^{(k-1)})'$. Hence if we define

$$\phi_k(\lambda) = C_{m,k}\lambda^{m-k} + C_{m-1,k}\alpha_1\lambda^{m-k-1} + \cdots \equiv \phi^{(k)}(\lambda)/k!$$

we may write (4) in the more useful form

$$\phi_1(d)b + \phi_2(d)b' + \cdots + \phi_m(d)b^{(m-1)} = 0. \tag{5}$$

We suppose now that $\phi_1(\lambda)\equiv\phi'(\lambda)$ is relatively prime to $\phi(\lambda)$.[4] Then $\phi_1(d)$ is a regular element in $\mathfrak{A}$. Hence if b is an element such that (5) holds,

$$b = -\phi_1(d)^{-1}\phi_2(d)b' - \cdots - \phi_1(d)^{-1}\phi_m(d)b^{(m-1)} = x'$$

where

$$x = -\phi_1(d)^{-1}\phi_2(d)b - \cdots - \phi_1(d)^{-1}\phi_m(d)b^{(m-2)}. \tag{6}$$

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[3] Loc. cit. footnote 1, p. 209.

[4] This condition will be satisfied if d generates a separable algebraic field over Φ.

This proves the following theorem.

THEOREM 1. *Let $\mathfrak{A}$ be an arbitrary algebra and let d be an algebraic element of $\mathfrak{A}$ having a minimum polynomial $\phi(\lambda)$ relatively prime to its derivative. Then* (5) *is a necessary and sufficient condition in order that the element b be a d-integral. When the condition holds, $b=x'$ where x is given by* (6).

We suppose now that $\mathfrak{A}$ is a simple algebra with an identity satisfying the descending chain condition for left (right) ideals. Then $\mathfrak{A}=\mathfrak{D}_h$, a matrix algebra of h rows over the (not necessarily finite) division algebra $\mathfrak{D}$ and conversely any algebra of this form satisfies our condition. As before let d be an algebraic element of $\mathfrak{A}$ and let $\phi(\lambda)$ be its minimum polynomial. Let b be an element of $\mathfrak{A}$ such that (5) holds. Then (4) holds and hence the minimum polynomial of v as well as of u is $\phi(\lambda)$. Since $d, b\in\mathfrak{A}=\mathfrak{D}_h$ the matrices u and $v\in\mathfrak{D}_{2h}$ and these may be regarded as the matrices of linear transformations in a $2h$-dimensional vector space $\mathfrak{R}$ over $\mathfrak{D}$. Let T be the linear transformation corresponding to v. Then according to the form of v we have an h-dimensional subspace $\mathfrak{S}$ of $\mathfrak{R}$ invariant under T such that the matrix of T in $\mathfrak{S}$ is d and the matrix of T in the difference space $\mathfrak{R}-\mathfrak{S}$ is also d.

We suppose now that $\phi(\lambda)$ is a product of irreducible factors in $\Phi[\lambda]$. In this case the linear transformation is completely reducible.[5] Hence there exists a subspace $\mathfrak{S}'$ invariant under T such that $\mathfrak{R}=\mathfrak{S}+\mathfrak{S}'$, $\mathfrak{S}\cap\mathfrak{S}'=0$ and such that the matrix of T in $\mathfrak{S}'$ is also d. Let $x_1, \cdots, x_h, x_{h+1}, \cdots, x_{2h}$ be the original basis of $\mathfrak{R}$ relative to which T has the matrix v so that $x_1, \cdots, x_h$ is a basis for $\mathfrak{S}$. Corresponding to the decomposition $\mathfrak{R}=\mathfrak{S}+\mathfrak{S}'$ we have the basis $x_1, \cdots, x_h, x'_{h+1}, \cdots, x'_{2h}$. The matrix relating this basis to the original one has the form

$$\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix},$$

where $p, q\in\mathfrak{D}_h$ and the matrix of T relative to the basis $x_1, \cdots, x_h$, $x'_{h+1}, \cdots, x'_{2h}$ is u. Hence we have the equation

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}^{-1} \begin{pmatrix} d & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}.$$

This implies that $dq=qd$ and that $bq=pd-dp$. Since the matrix

[5] See the author's paper *Pseudo-linear transformations*, Ann. of Math. vol. 38 (1937) p. 498.

$$\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}$$

is regular, q is regular and hence we have the relation $b=xd-dx$ where $x=pq^{-1}$.

THEOREM 2. *Let $\mathfrak{A}=\mathfrak{D}_h$ where $\mathfrak{D}$ is a division algebra over Φ and let d be an algebraic element of $\mathfrak{A}$. Then if the minimum polynomial $\phi(\lambda)$ of d is a product of distinct irreducible factors in $\Phi[\lambda]$, the condition* (5) *is necessary and sufficient in order that the element b of $\mathfrak{A}$ be a d-integral.*

We next let $\mathfrak{A}=\Phi_h$, the matrix algebra of h rows over Φ. Let d be a non-derogatory matrix in Φ_h. Thus d has only one invariant factor $\phi(\lambda)\neq 1$ and $\phi(\lambda)$ is the minimum polynomial of d. Let b be a matrix such that (5) holds and consider the matrix v as before. The minimum polynomial of v is $\phi(\lambda)$. If T is the linear transformation in the $2h$-dimensional space over Φ associated with the matrix v then $\mathfrak{R}$ contains an invariant subspace $\mathfrak{S}$ whose matrix is d. Since d is non-derogatory, $\mathfrak{S}$ is a cyclic subspace and its order is the minimum polynomial of T in $\mathfrak{R}$. Now it is known that this implies that $\mathfrak{R}=\mathfrak{S}+\mathfrak{S}'$, $\mathfrak{S}\cap\mathfrak{S}'=0$ where $\mathfrak{S}'$ is also invariant relative to T.[6] A repetition of the argument used to prove Theorem 2 will now yield the following theorem.

THEOREM 3. *Let d be a non-derogatory matrix in the matrix algebra Φ_h and let $\phi(\lambda)$ be its minimum polynomial. Then the condition* (5) *is necessary and sufficient that the matrix b be a d-integral.*

We give finally an example in which the condition (5) is not sufficient to insure that an element be a d-integral. Let

$$d=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the minimum polynomial of d is $\phi(\lambda)=\lambda^2$. Since $bd=db=0$, b satisfies (5). On the other hand, the invariant factors of the matrices u and v here are respectively $\lambda^2, \lambda^2, \lambda, \lambda$ and $\lambda^2, \lambda^2, \lambda^2$. It follows that these matrices are not similar and hence b is not a d-integral.

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[6] See van der Waerden's *Moderne Algebra*, vol. 2, pp. 129–130. The proof given there of this theorem for ordinary finite groups is also valid for vector spaces relative to a single linear transformation.

SYMBOLIC SOLUTION OF CERTAIN PROBLEMS IN PERMUTATIONS

IRVING KAPLANSKY

1. Introduction. It is our purpose in this paper to show how algebraic symbolism can be applied to the problem of enumerating certain types of restricted permutations. The method rests on a symbolic interpretation of the method of inclusion and exclusion, or rather its probability analogue, the formula of Poincaré; the use of the terminology of probability is a matter of convenience only, the considerations being purely combinatorial. The problems considered are of two kinds: those where the restriction of position of elements is absolute (§3, "card-matching" problems), and those where elements are restricted relative to others (§§5, 6). In §4 approximations to the symbolic formulas are obtained. Though these problems are rather special in character, the foundations of the method (§2) are quite general and should find other applications.

2. Symbolic expressions. Consider n events $A_1, \cdots, A_n$ and let $p(A_{i_1} \cdots A_{i_k})$ denote the probability of the joint occurrence of $A_{i_1} \cdots A_{i_k}$. The probability that none of $A_1, \cdots, A_n$ occurs, which we shall denote by P_0, is given by Poincaré's formula:

$$P_0 = 1 - \sum p(A_i) + \sum p(A_iA_j) \cdots . \tag{1}$$

In the case of complete symmetry, that is, where each $P(A_{i_1} \cdots A_{i_k})$ is a function ϕ_k of k alone,

$$P_0 = 1 - n\phi_1 + {}_nC_2\phi_2 - \cdots .$$

By using the displacement operator E defined by $E^k\phi_0 = \phi_k$, we may write more compactly $P_0 = (1-E)^n\phi_0$.

In the cases which we shall consider, however, this complete symmetry will be lacking and instead we shall have the following partial substitute, which we shall call *quasi-symmetry*: $p(A_{i_1} \cdots A_{i_k})$ is either equal to zero or to a function ϕ_k of k alone. To evaluate P_0 for such cases a symbolic device of Broderick [2][1] is helpful; we may write (1) in the form

$$P_0 = (1 - pA_1)(1 - pA_2) \cdots (1 - pA_n), \tag{2}$$

where the multiplication is symbolic in the sense that $p(A_i)p(A_j) \cdots$

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

is interpreted as $p(A_iA_j \cdots)$.[2] The following rule can then be stated.

To evaluate (2) *in the case of quasi-symmetry, first suppress all products that vanish; from the remaining expression form a polynomial* $f(E)$ *by replacing each term* $p(A_i)$ *by* E. *Then* P_0 *will equal* $f(E)\phi_0$.

The advantage of this point of view lies in the fact that in the computation of $f(E)$ we may resort to all the devices of formal algebra. A second advantage, perhaps more important, was pointed out by Fréchet [3]. Once $f(E)$ has been found, we have simultaneously solved the problem of finding P_r, the probability that exactly r of the events $A_1, \cdots, A_n$ occur. For Poincaré's formula gives

$$P_r = \sum_{k=r}^{n} (-1)^{k-r} {}_kC_{k-r} \sum p(A_{i_1} \cdots A_{i_k})$$

which can be rewritten

$$P_r = \sum_{k=0}^{n} (-1)^r {}_kC_r \sum (-1)^k p(A_{i_1} \cdots A_{i_k}). \tag{3}$$

Comparison of (1) and (3) shows that we may use the same polynomial $f(E)$ to evaluate P_r provided we replace ϕ_k by $(-1)^r {}_kC_r\phi_k$; that is,

$$P_r = f(E)\psi_0 \quad \text{where} \quad \psi_k = (-1)^r {}_kC_r\phi_k.$$

Moreover, since

$${}_kC_0 - {}_kC_1 + {}_kC_2 - \cdots + (-1)^r {}_kC_r = (-1)^r {}_{k-1}C_r,$$

the use of $(-1)^r {}_{k-1}C_r\phi_k$ will give the probability that *at most* r of the n events happen.

3. **Card-matching problems.**[3] A general version of the so-called "card-matching" problem can be formulated as follows: in a permutation of $1, \cdots, n$ what is the probability that the integer i is not in the jth place, also k is not lth, and so on? More generally we can ask for the probability that these conditions are violated precisely r times. For a convenient notation let p_{ij} denote the probability that i is jth; then we are to evaluate a product of the form

$$(1 - p_{ij})(1 - p_{kl}) \cdots .$$

We must first observe that the condition of quasi-symmetry is fulfilled; in fact the (symbolic) product of k of the p's is either 0 or

[2] We refrain from introducing a special notation for this symbolic multiplication, as it will always be clear from the context when it is intended.

[3] Cf. [1] for an extensive bibliography.

$(n-k)!/n!$. In accordance with our previous notation we shall therefore write

$$\phi_k = (n-k)!/n!, \qquad \psi_k = (-1)^r {}_kC_r(n-k)!/n! \tag{4}$$

in the remainder of the paper. The question of determining when a product of p_{ij}'s vanishes is readily answered; $p_{ij}\,p_{kl}=0$ if $i=k$ or $j=l$ and a product of several p's is 0 if and only if it contains such a pair. This observation makes it comparatively easy to compute $f(E)$ in some famous problems of the card-matching type.

In each of the problems that follows, the solution is provided by finding the polynomial $f(E)$ defined in §2; the probability that a permutation violates the conditions r times is $f(E)\psi_0$, with $\psi_k=E^k\psi_0$ given in (4).

(i) The classical "problème des rencontres" requires merely that i shall not be ith. We have to compute $(1-p_{11})\cdots(1-p_{nn})$. No products vanish so this is $(1-E)^n$.

(ii) In a widely known generalization of the preceding problem, the n integers are divided into n/a subsets of a each, and every integer is forbidden to appear in each of the a places originally occupied by the members of its subset. Thus, with $a=2$, we can suppose the elements so numbered that the subsets are (1, 2), (3, 4), $\cdots$, and the conditions read: neither 1 nor 2 is 1st or 2nd, neither 3 nor 4 is 3rd or 4th, and so on. We find $n/2$ groups of 4 factors, the first of which is

$$(1-p_{11})(1-p_{12})(1-p_{21})(1-p_{22}) = 1-p_{11}-p_{12}-p_{21}-p_{22}+p_{11}p_{22}+p_{12}p_{21}.$$

All vanishing products having now been eliminated, we have the answer $(1-4E+2E^2)^{n/2}$. For the usual deck of 52 cards a similar calculation yields

$$(1-16E+72E^2-96E^3+24E^4)^{13},$$

and for the 5×5 deck used in telepathic experiments:

$$(1-25E+200E^2-600E^3+600E^4-120E^5)^5.$$

We can generalize somewhat further by supposing that each of the a members of a subset are forbidden to appear in b, rather than a, places. We have then to compute

$$F(a,b) = \prod (1-p_{ij}) \qquad (i=1,\cdots,a;\ j=1,\cdots,b)$$

$$= \sum_{t=0}^{a} {}_aC_t\,{}_bC_t\,t!(-E)^t, \tag{5}$$

as follows from a simple combinatorial argument. We can even sup-

pose that the subsets are of different sizes $a_1, a_2, \cdots$ with the members forbidden $b_1, b_2, \cdots$ places respectively; this generalization leads to the polynomial $F(a_1, b_1)F(a_2, b_2)\cdots$. We remark again that by applying this operator to ψ_0, ψ defined as in (4), we obtain the probability that a permutation will violate the stated conditions precisely r times. This result was published without proof in [4].

(iii) In the "problème des ménages" the conditions read: 1 shall not be 1st or 2nd, 2 shall not be 2nd or 3rd, $\cdots$, n not nth or 1st. The product

$$(1 - p_{11})(1 - p_{12}) \cdots (1 - p_{nn})(1 - p_{n1})$$

was evaluated in [5] and found to be

$$U_n = \sum_{k=0}^{n} {}_{2n-k}C_k 2n(-E)^k/(2n-k). \tag{6}$$

But we are able further to solve Lucas' more difficult problem [6, p. 491] of finding the number of permutations "discordant" with any two given ones.[4] (The problème des ménages is the case where the given permutations are $1, 2, \cdots, n$ and $2, 3, \cdots, n, 1$.) The solution is as follows: considering one of the given permutations to be a permutation of the other, represent it as a product of disjoint cycles of length $a, b, c, \cdots$. Then the polynomial giving the probability of a permutation discordant with both is $U_aU_bU_c\cdots$, where U_n is given by (6) for $n \neq 1$ and $U_1 = 1 - E$.

(iv) As a final somewhat more intricate example, suppose $1, \cdots, n$ to be divided again into subsets of a each, and let every integer be forbidden to appear in the places occupied by all the *other* members of its subset—its own original position is permitted. The answer, which we state without proof, is

$$[F(a, a) + aEF(a-1, a-1) + {}_aC_2E^2F(a-2, a-2) + \cdots + E^a]^{n/a},$$

F being defined as in (5).

4. Asymptotic approximations. The symbolic formulas obtained in §3 provide the exact answers to the problems in question and are readily calculated for small values of n. For large values of n their exact evaluation is out of the question, but we can readily find approximations. The method is sufficiently well illustrated by a

[4] In an article submitted to the American Mathematical Monthly, John Riordan gives a detailed investigation of this solution, identifying it with one of Touchard [8], and applying it to the enumeration of n by 3 Latin rectangles.

consideration of the symbolic expression $(1-aE)^{zn}\phi_0$, where ϕ_k is defined as in (4) and zn is a positive integer not exceeding n.

$$(7) \qquad (1 - aE)^{zn}\phi_0 = \sum_{k=0}^{n} (-a)^k (zn)_k / k! n_k,$$

where $n_k = n(n-1)\cdots(n-k+1)$. Since

$$(8) \qquad (zn)_k / n_k = z^k - (k^2 - k)(z^{k-1} - z^k)n^{-1}/2 + O(n^{-2}),$$

it is natural to conclude from (7) and (8) that

$$(9) \qquad (1 - aE)^{zn}\phi_0 = e^{-az}[1 - a^2(z - z^2)n^{-1}/2 + O(n^{-2})].$$

However, since the error term of (8) involves k, a more detailed investigation is necessary.

Let $m = [n^{1/3}]$. We split the sum (7) into two parts: S_1, the sum from $k=0$ to $m-1$, and S_2, the sum from m to n. In all the applications $z \leqq 1$ and we shall treat only that case. Then $(zn)_k \leqq n_k$ and

$$(10) \qquad |S_2| \leqq \sum_{m}^{\infty} |a|^k / k! = O(m^{-6}) = O(n^{-2}).$$

To estimate S_1 the following lemma is convenient.

LEMMA. *Suppose $k \leqq m$ and let R denote the sum of the absolute values of the terms in n_k which are of degree not greater than $k-2$ in n. Then $R < k^6 n^{k-2}$.*

PROOF. The term in n^i is less than

$$n^i(1 + 2 + \cdots + k - 1)^{k-i} < n^i k^{2k-2i}.$$

Now $i \leqq k-2$ implies

$$i + (2k - 2i - 4)/3 \leqq k - 2.$$

Hence $n^i k^{2k-2i} \leqq k^4 n^{k-2}$. Since R contains $k-1$ terms all less than $k^4 n^{k-2}$, we have $R < k^5 n^{k-2}$.

Next we consider the expression

$$(11) \qquad Q = (zn)_k - n_k z^k + n_k(k^2 - k)(z^{k-1} - z^k)n^{-1}/2.$$

Since the terms in n^k and n^{k-1} cancel out, we may apply our lemma to estimate each of the three terms of Q. Remembering $z \leqq 1$, we find $|Q| < Ck^7 n^{k-2}$ where C depends only on z. We next note that $n^m(n-m)^{-m} \to 1$ as $n \to \infty$, so that $n^k(n-k)^{-k}$ has an upper bound independent of n and k. Hence

$$|Q/n_k| < Ck^7 n^{-2} n^k (n - k)^{-k} < C_1 k^7 n^{-2},$$

and finally

$$\sum_{0}^{n-1} (-a)^k Q/k! n_k = O(n^{-2}). \tag{12}$$

If we substitute in (12) the expression for Q given by (11) we verify that S_1 equals the right member of (9). Together with (10) this proves (9).

The same method, with more elaborate calculation, gives the general result (here again ψ_k is defined in (4), and zn is a positive integer not exceeding n)

$$(1 - aE + bE^2 \cdots)^{zn}\psi_0 = e^{-az}(az)^r(r!)^{-1}$$
$$\cdot [1 - (a^2 + 2b - a^2z)(r^2 - r - 2arz + a^2z^2)(2a^2zn)^{-1} + O(n^{-2})],$$

showing that the distribution of the number of "hits" obtained in the card-matching problems approaches the Poisson distribution with parameter az. One may of course compute as many more terms of the asymptotic expansion as desired. (The author has computed the next two terms, but as the result is complicated and unverified, it will not be stated.)

For example, with a deck of n cards divided into 4 suits, we found the probability of exactly r matches to be

$$(1 - 16E + 72E^2 \cdots)^{n/4}\psi_0$$
$$= e^{-4}4^r(r!)^{-1}[1 - 21(r^2 - 9r + 16)/8n] + O(n^{-2}).$$

The probability of no matches ($r=0$) is

$$(1 - 16E + 72E^2 \cdots)^{n/4}\phi_0 = e^{-4}(1 - 42n^{-1}) + O(n^{-2}).$$

However the approximation is still too crude for n as small as 52.

5. **The "n-kings problem."** There is a second type of restricted permutations to which our method can be applied. Let it be required to find the probability that, in a permutation of $1, \cdots, n$, the integer i does not immediately precede j, k does not immediately precede l, and so on. We shall denote the probability that i immediately precedes j by q_{ij}. The expression to be evaluated is then

$$(1 - q_{ij})(1 - q_{kl}) \cdots .$$

The fundamental requirement of quasi-symmetry (§2) is fulfilled; in fact, the product of k of the q's is either 0 or the same expression as in (4): $\phi_k = (n-k)!/n!$. Moreover it is again true that $q_{ij}q_{kl} = 0$ if $i = k$ or $j = l$; this time however we have the further complication $q_{ij}q_{ji} = 0$,

$q_{ij}q_{jk}q_{ki}=0$, and so on. With this modification, the calculations proceed as before.

The simplest problem of this kind requires merely that i shall never precede $i+1$. We find

$$(1 - q_{12})(1 - q_{22}) \cdots (1 - q_{n-1,n}) = (1 - E)^{n-1}.$$

A modification that suggests itself is to require further that $i+1$ shall not precede i. Or we can reformulate the question as a chess-board problem: in how many ways can n kings be placed on an n by n board, one on each row and column, so that no two attack each other? We recognize a first cousin (albeit a shabby relative) of the famous "n-queens problem."[5]

We have to evaluate

$$A_n = (1 - q_{12})(1 - q_{21})(1 - q_{23})(1 - q_{32}) \cdots (1 - q_{n-1,n})(1 - q_{n,n-1}).$$

Denote by B_n the result of deleting from A_n the last factor. Then

$$A_n = B_n(1 - q_{n,n-1}), \tag{13}$$

$$B_n = A_{n-1}(1 - q_{n-1,n}). \tag{14}$$

Further we find

$$A_{n-1}q_{n-1,n} = B_{n-1}(1 - q_{n-1,n-2})q_{n-1,n} = B_{n-1}q_{n-1,n}. \tag{15}$$

From (14) and (15):

$$B_n = A_{n-1} - B_{n-1}q_{n-1,n}. \tag{16}$$

On substituting (16) in (13) we get

$$A_n = B_n - A_{n-1}q_{n,n-1}. \tag{17}$$

Now in the product $B_{n-1}q_{n-1,n}$ it is readily checked that the latter factor does not enter into any vanishing products. Its effect therefore is simply to multiply the polynomial B_{n-1} by E. That is to say, $B_{n-1}q_{n-1,n}=EB_{n-1}$, whence we can rewrite (16) as

$$B_n = A_{n-1} - EB_{n-1}. \tag{18}$$

Moreover it is clear from symmetry that $A_{n-1}q_{n-1,n}=A_{n-1}q_{n,n-1}$. From (15) and (17) we therefore obtain

$$A_n = B_n - EB_{n-1}. \tag{19}$$

Elimination of A from (18) and (19) yields

[5] If there were a chess piece "*" which combined the moves of rook and king, the name "n-*" problem" would be more suitable than n-kings. It might be added that the author has failed in several efforts to solve the n-queens problem by these methods.

$$B_n = (1 - E)B_{n-1} - EB_{n-2}. \tag{20}$$

The solution of the difference equation (20), with initial values $B_1 = 1$, $B_2 = 1 - E$, is

$$B_n = (u^n - v^n)/(u - v)$$

where $u, v = (1 - E \pm (1 - 6E + E^2)^{1/2})/2$. A_n can be found from (19), and $A_n\phi_0$ is the solution of the n-kings problem. The first 3 terms are:

$$A_n = 1 - (2n - 2)E + (2n^2 - 8n + 8)E^2 \cdots .$$

Check for $n = 4$:

$$B_4 = u^3 + u^2v + uv^2 + v^3 = 1 - 5E + 5E^2 - E^3,$$
$$B_3 = u^2 + uv + v^2 = 1 - 3E + E^2,$$
$$A_4 = B_4 - EB_3 = 1 - 6E + 8E^2 - 2E^3,$$
$$A_4\phi_0 = (4! - 6 \cdot 3! + 8 \cdot 2! - 2 \cdot 1!)/4! = 2/4!,$$

corresponding to the two permissible arrangements 3142 and 2413.

6. **A problem of Netto.** In [7, p. 84], the following problem is proposed. Let the integers $1, \cdots, n$ be divided into n/a sets of a each. What is the probability that in a permutation there are nowhere two consecutive elements from the same subset? Thus for a deck of cards ($a = 4$), we require the probability of going through the deck and never encountering two consecutive cards of the same denomination (aces, deuces, and so on).

If the integers $1, \cdots, a$ are taken to form the first subset, we have to compute

$$V_a = \prod (1 - q_{ij}),$$

the product taken over all $i, j \leqq a$ with $i \neq j$; then the answer to Netto's problem will be $V_a^{n/a}$. Let W_a denote the result of deleting from V_a the $a - 1$ factors $(1 - q_{ia})$ $(i = 1, \cdots, a)$. Then

$$V_a = W_a(1 - q_{1a}) \cdots (1 - q_{a-1,a}) = W_a - (a - 1)W_aq_{a-1,a}.$$

Now in the product $W_aq_{a-1,a}$ the effect of multiplication by $q_{a-1,a}$ will be to delete from W_a the remaining terms of the form $q_{a-1,i}$ and also the term $q_{a,a-1}$. In the product that remains it is now merely a harmless change of notation to replace a by $a - 1$, after which precisely V_{a-1} is left. Hence

$$W_aq_{a-1,a} = EV_{a-1},$$
$$V_a = W_a - (a - 1)EV_{a-1}. \tag{21}$$

By a repetition of virtually the same argument we can show that

$$W_a = V_{a-1} - (a-1)EW_{a-1}. \tag{22}$$

By eliminating W from (21) and (22), we obtain

$$V_a = (1 + 2E - 2aE)V_{a-1} - (a-1)(a-2)E^2V_{a-2}.$$

The solution of this difference equation with initial values $V_1 = 1$, $V_2 = 1 - 2E$ is $F(a, a-1)$ in the notation of (5). Our final answer is thus $[F(a, a-1)]^{n/a}\phi_0$.

With $a = 4$ the result becomes

$$(1 - 12E + 36E^2 - 24E^2)^{n/4}\phi_0 = e^{-2} + O(n^{-1}).$$

The odds are about 19 to 1, although to the author's knowledge this game has been played at even money!

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ORBIT-CLOSURE DECOMPOSITIONS AND ALMOST PERIODIC PROPERTIES

W. H. GOTTSCHALK

Let X be a metric space with metric ρ, let $f(X) \subset X$ be a continuous mapping, and let $h(X) = X$ be a homeomorphism. For $x \in X$, the set $\sum_{n=0}^{+\infty} f^n(x)$ is called the *semi-orbit* of x under f and the set $n\sum_{-\infty}^{+\infty} h^n(x)$ is called the *orbit* of x under h. For $x \in X$, the closure of the semi-orbit of x under f is called the *semi-orbit-closure* of x under f and the closure of the orbit of x under h is called the *orbit-closure* of x under h.

A nonvacuous subset Y of X is said to be *semi-minimal* (*minimal*) under $f(h)$ provided that the semi-orbit-closure (orbit-closure) of each point of Y is Y. Clearly, any two semi-minimal (minimal) sets are either coincident or disjoint. It is easily proved that a subset Y of X is semi-minimal (minimal) under $f(h)$ if and only if Y is nonvacuous, closed, $f(Y) \subset Y(h(Y) = Y)$, and furthermore Y contains no proper subset with these properties. We follow Birkhoff [2, p. 198][1] in the terminology of "minimal set."

A *decomposition* of X is defined to be a collection of nonvacuous closed pairwise disjoint subsets of X which fill up X. We say that the mapping f *gives a semi-orbit-closure* (*a semi-minimal set*) *decomposition* provided that the collection of semi-orbit-closures (semi-minimal sets) is a decomposition of X. Also, it is said that the homeomorphism h *gives an orbit-closure* (*a minimal-set*) *decomposition* provided that the collection of orbit-closures (minimal sets) is a decomposition of X.

A point x of X is said to be *almost periodic* under f provided that to each $\epsilon > 0$ there corresponds a positive integer N with the property that in every set of N consecutive positive integers appears an integer n such that $\rho(x, f^n(x)) < \epsilon$. The mapping f is said to be *pointwise almost periodic* provided that each point of X is almost periodic under f. It is to be noted that various writers use the above terms in different senses and employ other terminologies for these notions.

LEMMA 1. *The mapping f (homeomorphism h) gives a semi-orbit-closure (an orbit-closure) decomposition if and only if $f(h)$ gives a semi-minimal-set (a minimal-set) decomposition; and in either event, the two decompositions coincide.*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

The proof is easy and will be omitted.

LEMMA 2. *In order that the homeomorphism h give an orbit-closure decomposition it is sufficient that h give a semi-orbit-closure decomposition; and in case X is compact, this condition is also necessary. In either event, the two decompositions coincide.*

PROOF. The proof of the sufficiency is easy and will be omitted. We establish the necessity. Let C be an orbit-closure. By Lemma 1, it is enough to show that C is a semi-minimal set. Let Y be a nonvacuous closed subset of C such that $h(Y) \subset Y$. The proof will be completed if we show $Y = C$. Define $Z = \prod_{n=0}^{+\infty} h^n(Y)$. Now Z is a nonvacuous closed subset of C such that $h(Z) = Z$. Since C is a minimal set by Lemma 1, $Z = C$ and, hence, $Y = C$.

LEMMA 3. *If $x \in X$ is almost periodic under f, then the semi-orbit-closure C of x is semi-minimal.*

PROOF. Suppose C is not semi-minimal. Then there exists a point y of C such that the semi-orbit-closure Y of y is not C. Now $Y \subset C$. Also $x \notin Y$, since otherwise $C \subset Y$ and thus $Y = C$. Let 2ϵ be the distance from x to Y. There exists a positive integer N such that in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x)) < \epsilon$. Choose $\delta > 0$ so small that $z \in X$ with $\rho(y, z) < \delta$ implies $\rho(f^i(y), f^i(z)) < \epsilon$ $(i = 1, 2, \cdots, N)$. Now there exists a nonnegative integer p such that $\rho(y, f^p(x)) < \delta$. Also it is possible to find an integer q, $1 \leqq q \leqq N$, so that $\rho(x, f^{p+q}(x)) < \epsilon$. Hence, $\rho(x, f^q(y)) < 2\epsilon$ which is impossible.

LEMMA 4. *If X is locally compact and if the subset Y of X is semi-minimal under f, then each point of Y is almost periodic.*

PROOF. Assume some point x of Y is not almost periodic. There exists a neighborhood U of x such that $\overline{U}$ is compact and such that for each positive integer n there exists a point x_n of $U \cdot Y$ with the property that $f^m(x_n) \notin U$ $(m = 1, 2, \cdots, n)$. Some subsequence of $x_1, x_2, \cdots$ converges to some point, say y, of $\overline{U} \cdot Y$. There exists a positive integer M such that $f^M(y) \in U$ and, hence, there also exists a neighborhood V of y such that $f^M(V) \subset U$. Choose an integer N so that $N > M$ and $x_N \in V$. Then, $f^M(x_N) \in U$ which is a contradiction.

THEOREM 1. *In order that the mapping f give a semi-orbit-closure decomposition, it is sufficient that f be pointwise almost periodic; and in case X is locally compact, this condition is also necessary.*

The proof follows easily from Lemmas 1, 3 and 4.

THEOREM 2. *In order that the homeomorphism h give an orbit-closure decomposition it is sufficient that h be pointwise almost periodic; and in case X is compact, this condition is also necessary.*

The proof proceeds easily from Theorem 1 and Lemma 2.

The preceding results are related to certain results of Birkhoff [2, p. 199] and Hall and Kelley [4]. Gottschalk [3] contains a direct proof of Theorem 2.

We pause in our main development to comment on the rôle of local compactness in the second part of Theorem 1. In this case the semi-orbit-closures are actually compact, as the following indicates.

THEOREM A. *If $x \in X$ is almost periodic under f and if there exists a neighborhood U of x whose closure is compact, then the semi-orbit-closure of x is itself compact.*

Proof. There exists an integer N and a sequence $\{n_i \mid i=0, 1, \cdots\}$ of integers such that $0=n_0<n_1<\cdots$, $n_{i+1}-n_i \leqq N$ and $f^{n_i}(x) \in U$ for all i. Define $K=\sum_{r=0}^{N} f^r(\overline{U})$. Now K is compact. We show the semi-orbit of x is contained in K, which completes the proof. Let n be any non-negative integer. There exists a non-negative integer i such that $n_i \leqq n \leqq n_{i+1}$. Hence, $f^n(x)=f^{n-n_i}f^{n_i}(x) \in f^{n-n_i}(U) \subset K$.

THEOREM B. *If $Y \subset X$ is semi-minimal under f and if Y intersects a neighborhood U whose closure is compact (in particular, if X is locally compact), then Y is itself compact.*

Proof. Let $y \in Y \cdot U$. By the argument used in the proof of Lemma 4, it can be shown that y is almost periodic. The conclusion now follows from Theorem A. (Theorem B is not valid for a minimal set as the example of a discrete infinite orbit shows.)

Besicovitch [1] has constructed an interesting example of a homeomorphism of the plane onto itself which possesses *some* semi-orbits dense in the plane and which leaves the origin fixed. He seems to remark at the end of his paper (p. 65) that *every* semi-orbit, excluding the origin, is also dense in the plane. Theorem B would indicate that either this remark or our interpretation of it is in error. For, take $X=Y$ to be the plane with the origin deleted. If every semi-orbit is dense in the punctured plane, then the punctured plane would be compact. *Question: In Besicovitch's example* [1], *is the orbit of every point of the plane, excepting the origin, dense in the plane?* We now continue with the main sequence of theorems.

Let $\{X_n \mid n=1, 2, \cdots\}$ be a sequence of subsets of X. The set of all points x of X such that each neighborhood of x intersects X_n for

almost all (infinitely many) positive integers n is denoted by $\liminf \{X_n\}$ ($\limsup \{X_n\}$). In case $\liminf \{X_n\} = \limsup \{X_n\}$, we denote this set by $\lim \{X_n\}$. Of course, for any sequence $\{X_n\}$, $\liminf \{X_n\} \subset \limsup \{X_n\}$.

Let D be a decomposition of X. For $x \in X$, let $D(x)$ denote the element of D containing x. The decomposition D is said to be *continuous* provided that $x_0, x_n \in X$ $(n=1, 2, \cdots)$ with $x_n \to x_0$ implies $D(x_0) \subset \liminf \{D(x_n)\} \subset \limsup \{D(x_n)\} \subset D(x_0)$.

Following G. A. Hedlund, we say the mapping f is *uniformly pointwise almost periodic* provided that to each $\epsilon > 0$ there corresponds a positive integer N such that if $x \in X$, then in every set of N consecutive positive integers appears an integer n so that $\rho(x, f^n(x)) < \epsilon$. Clearly, if f is uniformly pointwise almost periodic, then f is pointwise almost periodic.

LEMMA 5. *For $x \in X$, let $C(x)$ denote the semi-orbit-closure of x under f. If $x_0, x_n \in X$ $(n=1, 2, \cdots)$ with $x_n \to x_0$, then $C(x_0) \subset \liminf \{C(x_n)\}$.*

PROOF. Let $x \in C(x_0)$ and let U be any neighborhood of x. For some non-negative integer k, $f^k(x_0) \in U$. By the continuity of f^k, $f^k(x_n) \in U$ for almost all positive integers n, that is, U intersects $C(x_n)$ for almost all positive integers n. The conclusion follows.

THEOREM 3. *In order that the mapping f give a continuous semi-orbit-closure decomposition it is sufficient that f be uniformly pointwise almost periodic; and in case X is compact, this condition is also necessary.*

PROOF. We establish the sufficiency. Let D denote the collection of semi-orbit-closures. By Theorem 1, D is a decomposition of X. By Lemma 5, it is enough to prove that $\limsup \{C(x_n)\} \subset C(x_0)$ for x_0, $x_n \in X$ $(n=1, 2, \cdots)$ with $x_n \to x_0$, where $C(x)$ denotes the semi-orbit-closure of the point x. Assume this is false. Then there exist points x_0, $x_n \in X$ $(n=1, 2, \cdots)$ such that $x_n \to x_0$ and $\limsup \{C(x_n)\} \not\subset C(x_0)$. Thus there exist a point x of X, a monotone increasing sequence $n_1, n_2, \cdots$ of positive integers, and a sequence $m_1, m_2, \cdots$ of non-negative integers such that $f^{m_i}(x_{n_i}) \to x$ and $x \notin C(x_0)$. Hence, $C(x) \cdot C(x_0) = \Lambda$ and $x_0 \notin C(x)$. Let 2ϵ denote the distance from x_0 to $C(x)$. Since f is uniformly pointwise almost periodic, there exists a positive integer k such that for each positive integer i it is possible to find an integer k_i with the properties that $1 \leqq k_i \leqq k$ and $\rho(x_{n_i}, f^{m_i+k_i}(x_{n_i})) < \epsilon$. There exists an integer k_0 such that $k_i = k_0$ for infinitely many positive integers i. Since also $x_{n_i} \to x_0$ and $f^{m_i+k_0}(x_{n_i}) \to f^{k_0}(x)$, we have $\rho(x_0, f^{k_0}(x)) \leqq \epsilon < 2\epsilon$. Hence, the distance from x_0 to $C(x)$ is less than 2ϵ. This is a contradiction.

We establish the necessity. Suppose f is not uniformly pointwise almost periodic. Then there exists a positive number ϵ such that for each positive integer n it is possible to find a positive integer m_n and a point x_n of X such that $N_\epsilon(x_n)\cdot\sum_{i=0}^{n} f^{m_n+i}(x_n)=\Lambda$, where $N_\delta(y)$ denotes the δ-neighborhood of the point y. We may suppose that $x_n\to x_0$ and $f^{m_n}(x_n)\to x$ for some points x_0, x of X. For all sufficiently large positive integers n, $N_{\epsilon/2}(x_0)\cdot\sum_{i=0}^{n} f^{m_n+i}(x_n)=\Lambda$. It follows that $N_{\epsilon/2}(x_0)\cdot C(x)=\Lambda$ and $x_0\notin C(x)$, where $C(y)$ denotes the semi-orbit-closure of the point y. Now $\lim\{C(x_n)\}=C(x_0)$ and $C(x)=\lim\{C(f^{m_n}(x_n))\}=\lim\{C(x_n)\}$. Hence, $C(x_0)=C(x)$ and $x_0\in C(x)$. This is a contradiction.

THEOREM 4. *In order that the homeomorphism h give a continuous orbit-closure decomposition it is sufficient that h be uniformly pointwise almost periodic; and in case X is compact, this condition is also necessary.*

The proof follows readily from Theorem 3 and Lemma 2.

It is worthy of note that if X is compact, then there exists a subset Y of X such that $f(Y)=Y$ is uniformly pointwise almost periodic. The proof is short. The property P of being a nonvacuous closed subset Z of X such that $f(Z)\subset Z$ is easily shown to be inducible. By the Brouwer reduction theorem, there exists a subset Y which has property P irreducibly. Then Y is semi-minimal and $f(Y)=Y$, since $f(Y)$ has property P. The conclusion now follows from Theorem 3.

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ON CONTINUA WHOSE LINKS ARE NON-INTERSECTING

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There are many examples known of compact continua no two of whose simple links intersect, even though uncountably many of the links are nondegenerate. On page 346 of his *Foundations of point set theory* [1],[1] R. L. Moore has given an example of a compact continuum which is the sum of an uncountable collection of nondegenerate simple links, only a countable number of which intersect. In view of these results it seems natural to inquire whether there is a compact continuum which is the sum of an uncountable collection of mutually exclusive nondegenerate simple links. Theorem 1 gives a negative answer. If we do not require that the continuum be compact, but do require that it be a separable locally connected Moore space,[2] then Theorem 7 shows that the answer is again negative.

References to theorem and page numbers refer to Moore's book, to which the reader is directed for definitions not given here.

THEOREM 1.[3] *If M is a compact metric continuum which is not a simple link of itself and no two of its simple links intersect, then uncountably many are degenerate, and the set which is the sum of the nondegenerate simple links of M is the sum of a countable number of closed point sets.*[4]

LEMMA. *Under the hypothesis of Theorem* 1, *the simple links of M form an upper semi-continuous collection, G, filling up M such that G is an acyclic continuous curve every interval of which contains a degenerate element of G.*

PROOF. By Theorem 93, p. 67, each simple link of M is a continuum. Let P be the sequential limiting point of a sequence of points of distinct links $L_1, L_2, L_3, \cdots$ of M. Let Q be any point of the limiting set of $L_1, L_2, L_3, \cdots$. Clearly no point separates P from Q

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

[2] A space is said to be a Moore space if it satisfies Axiom 0 and the first three parts of Axiom 1 of Moore's book.

[3] In the terminology of transformations, Theorem 1 is equivalent to the following: Let f be a monotone mapping of the compact continuum M into a dendrite N. If for each point X of N no point of M separates any two points of $f^{-1}(X)$ and for each two points X and Y of N some point of M separates $f^{-1}(X)$ from $f^{-1}(Y)$, then for uncountably many points X of N the set $f^{-1}(X)$ is degenerate.

[4] The proof of this theorem was altered after refereeing.

in M, and since by Theorem 103, p. 71, no point of M is an improper point of M, it follows that the collection G is upper semi-continuous. By Theorem 23(c), p. 344, G is a dendrite with respect to its elements. Suppose that the interval AB of G contains no degenerate element. Then the set, K, of all points each of which separates A from B in M is countable, for no link can contain more than two points of K, and by Theorem 24(a), p. 350, no more than a countable number of nondegenerate links can contain cut points of M. However, the collection, T, of all simple links of M in AB that intersect K is dense in AB. From Theorem 24, p. 348, we have that G contains an uncountable continuous subcollection, C, dense in AB.[5] Let P be a point of K contained in a link L, and let M' denote the subcontinuum of M consisting of the sum of the elements of AB. Then $M'-P$ is the sum of two mutually separated sets, H_1 and H_2, containing A and B respectively; and since L separates AB into two connected sets of elements of G, $M'-P=L_1+L_2+(L-P)$, where L_1 and L_2 are mutually separated connected sets not intersecting L. If $L-P$ contains limit points of both L_1 and L_2, then it intersects both K_1 and K_2, which contradicts Theorem 98, p. 70. Hence $L-P$ and H_1, say, are mutually separated. But since C is dense in AB it follows that there is a sequence $L_1, L_2, L_3, \cdots$ of elements of C whose limiting set (in M) is P. Since G is upper semi-continuous, as $n\to\infty$, diam $L_n\to 0$. Let H be an element of C, and let d be its diameter. Every open segment of AB containing H also contains elements of T, and therefore contains elements of C of diameter less than $d/2$, which contradicts the continuity of C. Hence AB contains a degenerate link of M.

PROOF OF THEOREM 1. By the lemma and Theorem 5.1, p. 131, of [4], the collection G' of all nondegenerate elements of G is semi-closed, and each element of G' is a component of G'^*. By Theorem 5.41, p. 132, of [4], G'^* is the sum of a countable number of closed point sets. If the degenerate elements of G were countable in number, the decomposition space G would be the sum of a countable number of closed sets, no one of which contains a domain, which would contradict Theorem 15, p. 11.

A well known example due to Anna M. Mullikin [2], and described on p. 366 of Moore's book, of a connected set in the plane which is the sum of a countable number of mutually exclusive arcs, can be modified by replacing each arc by a simple closed curve to show that Theorem 1 is not true if the hypothesis of compactness is replaced by separability, even if each link is a compact continuum.

Another argument for Theorem 1 could be given by proving that

[5] Theorem 24 does not state that C is dense in AB, but this readily follows here.

under the hypothesis of Theorem 1 every simple link of M is a set $M(P)$.[6] Then Theorem 1 would follow from Theorem 2, which has some interest, due to the connection between dendratomic subsets, simple links, and sets $M(P)$ shown by Moore in chap. 5 of his book.

THEOREM 2. *If the compact metric continuum M is not an $M(P)$, then uncountably many sets $M(P)$ of M are degenerate.*

PROOF. By Theorem 23(a), p. 343, the collection, G, of all sets $M(P)$ of M is upper semi-continuous. By Theorem 24(a), p. 350, no more than a countable number of nondegenerate elements of G contain cut points of M. By hypothesis M has uncountably many cut points. If the theorem were false some element of G would contain uncountably many cut points of M, which contradicts Theorem 23(b), p. 344.

Since every simple link is a subset of some set $M(P)$, this theorem says that if the simple links are allowed to intersect in certain groups so that no two points in any group are separated by more than a countable number of points, but so that any two points in different groups are separated by uncountably many points, then the conclusion of Theorem 1 still holds.

Locally connected continua. If to the hypothesis of Theorem 1 we add the condition that M be locally connected, our theorem becomes trivial, since the nondegenerate simple links of a compact continuous curve are its true cyclic elements, and there are only a countable number of these. If we also remove the condition of compactness, then the problem becomes more interesting, and indeed the theorem would not be true without the assumption of separability. However, if M were assumed to be locally connected, separable, and complete, the theory of almost cyclic elements developed by F. B. Jones [3] would apply and our theorem could readily be proved, in virtue of Theorem 19 of his paper. Failure to assume completeness creates more difficulties, since arcwise connectivity would not be available. The theorem remains true, however, and the proof is not difficult. We prove first several theorems concerning simple links.

In the next four theorems, M will denote a locally connected, connected Moore space.

THEOREM 3.[7] *If L is a simple link of M, and R is a connected domain intersecting L, then $L \cdot R$ is connected. Hence L is connected and locally connected.*

[6] If P is a point of M, by $M(P)$ is meant the set of all points of M which are not separated from P by each of an uncountable set of points.

[7] Cf. Theorem 66, p. 147, of Moore, and Theorems 8 and 20 of Jones' paper.

PROOF. Suppose that $L \cdot R$ is the sum of two mutually separated sets, H and K. Some component, C, of $R-R \cdot L$ has a limit point, P, in H, and a limit point, Q, in K, as may be seen with the use of Theorem 2(b), p. 91. Let X be any point of C. No point of L separates P or Q from X in R. But by Theorem 12, p. 97,[8] some point Y separates X from $P+Q$ in R and is not itself separated from $P+Q$ by any point. Let Z be a proper point of M belonging to L. Some point, W, separates Y from Z in M, and W can be neither P nor Q. But then W cannot belong to L, and hence separates $P+Q$ from Y, which is impossible.

If R is taken to be M, it follows that L is connected.

THEOREM 4.[9] *If M is the sum of its simple links, and no two simple links of M intersect, then every improper point belongs to a nondegenerate continuum of improper points, no two lying in the same simple link.*

PROOF. Let X be an improper point of M. Let H be a point set containing X such that no two points of H belong to the same simple link of M or are separated by any point, and which is maximal with respect to this property. Since the simple links of M are mutually exclusive, H is nondegenerate. Also, every point of H is an improper point. Further, H is closed, for if the point Y of $M-H$ is a limit point of H, then no point separates Y from any point of H. Hence Y belongs to the same simple link as some point Z of H. But then Z separates Y from $H-Z$, which is impossible.

Suppose that H is the sum of two mutually exclusive closed sets, H_1 and H_2. Paralleling the argument of Theorem 3, we show the existence of a point Q of $M-H$ and two points, P_1 and P_2, belonging to H_1 and H_2 respectively, such that no point separates Q from both P_1 and P_2. Hence no point separates Q from any point of H. Further, Q is not in the same simple link as any point of H. Therefore Q belongs to H, which is a contradiction, and completes our proof.

THEOREM 5. *If M is the sum of its simple links and is not a simple link of itself, and the simple links of M are non-intersecting, they form an uncountable collection.*

PROOF. Suppose that the theorem is false. Then every point of M is a proper point of M, for if P were an improper point of M, by Theorem 4, P would lie in a nondegenerate continuum H of points, no two of which lie in the same simple link of M. But H must be

[8] See remarks on p. 465 of Moore concerning this theorem.

[9] This theorem should be contrasted with Theorem 99, p. 71.

countable, and no nondegenerate countable set is connected.[10] Hence, if A and B belong to distinct simple links of M, the set $I(A, B)$ of all points separating A from B is non-empty, and indeed any two points of $I(A, B)+A+B$ belonging to distinct links are separated by some point. Since no link can contain more than two points separating A from B, $I(A, B)$ is countable. It is then easy to construct two sequences of points, $P_1, P_2, P_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ such that (1) for each positive integer i, P_{i+1} precedes P_i, Q_{i+1} follows Q_i, and P_i follows Q_i in the order from A to B in $I(A, B)$; and (2) for any point X of $I(A, B)$ there is an i such that X is not between P_i and Q_i. But $P_1+P_2+P_3+\cdots$ cannot then have a limit point in $I(A, B)$, which contradicts Theorem 4.2, p. 51, of [4], the proof of which is valid in this space.

The example given following Theorem 1 shows that this theorem is not true without local connectivity.

The condition that M is the sum of its simple links in the hypothesis of this theorem is necessary, for there exist non-compact continuous curves which do not contain any simple links whatsoever. We give a rough description of one such: Jones has given on p. 781 of [3] an example of a space, H_1, containing a simple closed curve, C, each of whose points is an improper point, and each of whose simple links is a simple closed curve, $C_{1,\alpha}$, having only a point in common with C. By applying Jones' construction to each set $C_{1,\alpha}$, we can construct a space H_2 in which every point of H_1 is an improper point, and whose simple links are simple closed curves, $C_{2,\alpha}$, having only one point in common with H_1. Repeating Jones' construction again on the sets $C_{2,\alpha}$, we obtain a space H_3 in which every point of H_2 is an improper point. Continue this process indefinitely. The space H which is the limit of $H_1, H_2, H_3, \cdots$ is such that every point is an improper point, and hence H has no simple links at all.

THEOREM 6. *If M is separable, the nondegenerate simple links of M orm a countable collection, and each nondegenerate link is separable.*

PROOF. By hypothesis, M contains a countable dense subset, H. By Theorem 12, p. 97, each nondegenerate simple link contains a cut point of M. Suppose that the collection, G, of all nondegenerate simple links of M is uncountable. Either (1) there is a point, P, common to uncountably many elements of G, or (2) with Theorem 72, p. 50, and Theorem 68, p. 47, there can be shown to exist an uncountable subcollection, G', of G, two points, A and B, and a set, K, such

[10] Since a countable Moore space is completely separable, and hence metric, it cannot be connected, if it is nondegenerate.

that each element of G' contains one and only one point of K, and such that each point of K separates A from B, and belongs to an element of G'. In the last case, each element, L, of G' is intersected by a connected domain, D_L, whose closure does not intersect $I(A, B)+A+B$ (which by Theorem 4.2, p. 51, of [4] is closed and compact). There is a point, X, of H which is common to uncountably many of the domains D_L. Let G'' denote the collection of all links, L, of G' such that D_L contains X, and let K' denote the set of all points of K intersecting links of G''. Some point, Y, is a point of condensation of K', and there is a connected domain, D, containing Y, such that for uncountably many elements L of G'', $D_L \cdot D = 0$. But then no point separates a point of an element of G'' from any point of another element of G'', which is impossible. A similar argument shows that the first case is likewise impossible, P taking the role of Y.

We show that each simple link is separable. Let L be any nondegenerate link of M, and let H be a countable dense subset of M. For each point X of $H-H \cdot M$, there is a point, P_X, of L which separates X from some point of L. The set, H', consisting of $H \cdot L$ and all points P_X is countable. If D is a connected domain intersecting L, and Y is a point of $H \cdot D$, either Y is in L or P_Y is in D. In either case, D intersects H'. Hence H' is a countable dense subset of L.

The last part of the theorem is false without local connectivity.

THEOREM 7. *If M is separable, and is not a simple link of itself, but is the sum of its simple links, and no two links of M intersect, then uncountably many are degenerate.*

PROOF. This follows immediately from Theorems 5 and 6.

The example of Jones referred to following Theorem 4 shows that the condition of separability in Theorem 7 cannot be removed. A simple modification of this example gives a similar example where in addition the almost cyclic elements are mutually exclusive.

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PURDUE UNIVERSITY

ON THE QUADRICS ASSOCIATED WITH A POINT OF A SURFACE

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It is well known that among the three-parameter family of quadrics having a contact of the second order with an analytic surface at a point there are two systems of quadrics known as the quadrics of Darboux and the quadrics of Moutard. The particular quadric of the first family introduced in this note is of some interest. We propose to give a new geometrical construction for the cone of Segre by using Moutard quadrics and study the canonical pencil of lines associated with the surface.

Let $x^i(u, v)$ $(i=1, \cdots, 4)$ be the projective normal coordinates of a point on a surface S referred to its asymptotic net (u, v); then the functions x are solutions of a completely integrable system of differential equations in Fubini's canonical form

$$\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + p_{11} x, \\ x_{vv} &= \gamma x_u + \theta_v x_v + p_{22} x, \\ \theta &= \log \beta\gamma. \end{aligned} \tag{1}$$

As usual we introduce nonhomogeneous local coordinates of a point with respect to the tetrahedron $(xx_u x_v x_{uv})$, so that the equation of any quadric of Darboux is found to be

$$z - xy + kz^2 = 0, \tag{2}$$

where k denotes a parameter, while the equation of the quadric of Moutard which belongs to the tangent $y - nx = 0, z = 0$ is

$$\begin{aligned} &[4(\beta + \gamma n^3)^2 + 3n(\beta_u + 4\beta_v n + 4\gamma_u n^3 + \gamma_v n^4)]z^2 \\ &\quad + 36n^2 \left[z - xy + \frac{z^2}{2} \frac{\partial^2 \log \beta\gamma}{\partial u \partial v} \right] \\ &\quad - 12n(\beta - 2\gamma n^3) \left(y + \frac{1}{2} \frac{\partial \log \beta\gamma}{\partial u} z \right) z \\ &\quad - 12n^2(\gamma n^3 - 2\beta) \left(x + \frac{1}{2} \frac{\partial \log \beta\gamma}{\partial v} z \right) z = 0. \end{aligned} \tag{3}$$

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These quadrics intersect each other in the asymptotic tangents and a conic in the plane

$$z\left[4(\beta+\gamma n^3)^2+3n(\beta_u+4\beta_v n+4\gamma_u n^3+\gamma_v n^4)+36n^3K\beta\gamma\right.$$
$$\left.-6n\frac{\partial\log\beta\gamma}{\partial u}(\beta-2\gamma n^3)-6n^2\frac{\partial\log\beta\gamma}{\partial v}(\gamma n^3-2\beta)\right]$$
$$-12n(\beta-2\gamma n^3)y-12n^2(\gamma n^3-2\beta)x=0,$$

where we have placed

$$k=\frac{1}{2}\frac{\partial^2\log\beta\gamma}{\partial u\partial v}-K\beta\gamma. \tag{4}$$

The plane coordinates of this plane are evidently

$$\begin{aligned}&u_0=0,\\ &\rho u_1=-12n^2(\gamma n^3-2\beta),\qquad \rho u_2=-12n(\beta-2\gamma n^3),\\ &\rho u_3=4\beta^2-3\beta\phi+12\beta n^2\psi+(8\beta\gamma+36K\beta\gamma)n^3+12\gamma\phi n^4\\ &\quad-3\gamma n^5\psi+4\gamma^2n^6,\end{aligned} \tag{5}$$

where $\phi=\partial\log\beta\gamma^2/\partial u$ and $\psi=\partial\log\beta^2\gamma/\partial v$, so that the cone enveloped by it may be obtained by eliminating n from the equations

$$\begin{aligned}&\pi n^3+\omega n^2+\iota n+s=0,\\ &\sigma n^3+\tau n+\omega n+p=0,\end{aligned} \tag{6}$$

where

$$\begin{aligned}&\omega=7u_1\psi/2+7u_2\phi/2-6u_3,\\ &\pi=u_2\psi-2\gamma u_1,\qquad p=u_1\phi-2\beta u_2,\\ &s=(4+6K)\beta u_1,\qquad \sigma=(4+6K)\gamma u_2,\\ &\iota=u_1\phi+6(1+2K)\beta u_2,\qquad \tau=u_2\psi+6(1+2K)\gamma u_1.\end{aligned}$$

The result of carrying out the computation for a given value of K is

$$\begin{vmatrix}\pi & \omega & \iota & s\\ \sigma & \tau & \omega & p\\ 0 & p\pi-\sigma s & \omega p-\tau s & p\iota-\omega s\\ \sigma\omega-\pi\tau & \sigma\iota-\pi\omega & \sigma s-\pi p & 0\end{vmatrix}=0. \tag{7}$$

It should be remarked that when and only when $K=-2/3$ the equation of the cone enveloped by the planes (5) is different from (7),

since the latter in this case becomes an identity. From (2) and (4) we may infer naturally that *in the pencil of Darboux the quadric*

$$z - xy + \left(\frac{1}{2}\frac{\partial^2 \log \beta\gamma}{\partial u \partial v} + \frac{2}{3}\beta\gamma\right)z^2 = 0 \tag{8}$$

differs from other quadrics by a certain geometrical characterization.

The third polar of $z=0$ with respect to the cone (7) is

$$6u_1u_2u_3 - 3(u_1\psi + u_2\phi)u_1u_2 - (\beta u_2^2 + \gamma u_1^2) = 0, \tag{9}$$

which does not contain K. That is to say, *all the cones* (7) *with different values of* K *have the third polar of* $z=0$ *in common.*

As to the second polar of the same plane $z=0$ we have

$$[\cdots]u_1u_2 + u_1u_2\omega_2^2 + 6[(u_1\psi + u_2\phi)u_1u_2 - 2(\beta u_2^3 + \gamma u_1^3)] = 0 \tag{10}$$

where $[\cdots]$ does not contain ω. *Hence through each tangent of the surface* S *at* P *there are besides* $z=0$ *two tangent planes* p_1 *and* p_2 *of the cone* (10) *and the harmonic conjugate plane of* $z=0$ *with respect to* p_1 *and* p_2 *just envelopes the cone* (9). Thus we obtain a correspondence between a tangent of the surface and a plane through it. Especially, *the three corresponding planes of the tangents of Darboux are concurrent in the first directrix of Wilczynski.*

The equation of the cone (9) in point coordinates is found to be

$$\begin{aligned} &-3\left(y+\frac{\phi}{2}z\right)^2\left(x+\frac{\psi}{2}z\right)^2 + 2\gamma z\left(y+\frac{\phi}{2}z\right)^3 \\ &+ 2\beta z\left(x+\frac{\psi}{2}z\right)^3 - \frac{3}{2}\beta\gamma\left(x+\frac{\psi}{2}z\right)\left(y+\frac{\phi}{2}z\right)z^2 \\ &+ \frac{\beta^2\gamma^2}{16}z^4 = 0. \end{aligned} \tag{11}$$

It follows that $z=0$ *is the only double tangent plane along the two asymptotic tangents of* S *and therefore that there are three cuspidal generators, each passing through one of the points*

$$\left(0, \frac{\omega^2}{2}(\beta\gamma^2)^{1/3} - \frac{\psi}{2}, \frac{\omega}{2}(\gamma^2\beta)^{1/3} - \frac{\phi}{2}, 1\right), \qquad (\omega^3 = 1, \omega \neq 1).$$

The harmonic conjugate line of $z=0$ *with respect to the trihedral of the three cuspidal generators is also the first directrix of Wilczynski. Furthermore the three cuspidal tangent planes of the cone* (9) *are concurrent in the same directrix.*

It is worth noticing that the cones (10) with variable K have a common tangent plane which passes through each asymptotic tangent. The two planes thus obtained intersect in the canonical line $C(-7/12)$.[2]

Let us now attend to the particular quadric (8). Eliminating n from the equations

$$\pi n^2 + \omega n + p = 0,$$
$$\beta u_1(1 - 2(\gamma/\beta)n^2) = n(\gamma n^2 - 2\beta)u_2,$$

we obtain that the cone in this case is

$$(12)\qquad \begin{vmatrix} p\gamma u_2 & 2\gamma u_1 p & \pi\beta u_1 & \omega\beta u_1 - 2\beta u_2 p \\ -2\gamma\pi u_1 + \gamma u_2\omega & \gamma u_2 p & 2\beta\pi u_2 & \beta u_1\pi \\ \pi & \omega & p & 0 \\ 0 & \pi & \omega & p \end{vmatrix} = 0.$$

The second polar of the plane $z=0$ with respect to this cone is

$$(13)\qquad -u_1u_2\omega^2 + (u_1^2\pi + u_2^2 p)\omega = 0.$$

Therefore the quadric (8) *is distinguished from other quadrics in the pencil of Darboux by the fact that the corresponding cone* (12) *decomposes into a cone of the third class and a pencil of planes with* $C(-7/12)$ *as its axis.* Let p_1 and p_2 be the two tangent planes of the cone (12) through a given tangent of S at P but distinct from $z=0$, and t a plane through the same tangent such that

$$(z = 0, t; p_1, p_2) = \tau = \text{const.}$$

Then the equation of t is easily found to be

$$(14)\qquad (1-\tau)(u_1x + u_2y) + z\left[\frac{7}{12}(\psi u_1 + \phi u_2) - 5\tau(\psi u_1 + \phi u_2) - \frac{\tau}{3}\left(\frac{\gamma u_1^2}{u_2} + \frac{\beta u_2^2}{u_1}\right)\right] = 0,$$

u_1/u_2 being a parameter. *This plane envelopes a cone of the fourth order with three cuspidal tangent planes concurrent in the canonical line* $C(-(7-5K)/12(1-K))$, *while the latter is also the polar line of* $z=0$ *with respect to the trihedral of the three cuspidal generators.*

Finally, we shall give a geometrical interpretation to the cone of

[2] This line appeared in Sannia's paper. See Rendiconti R. Accademia dei Lincei (6) vol. 9 (1929) pp. 1081–1085.

Segre. Consider the two quadrics of Moutard belonging to the tangents t and t' at a point of the surface; they intersect in the asymptotic tangents and a conic. The limiting position of the plane of this conic as $t' \to t$ is

$$z[4\beta^2 - 4\gamma^2\eta^6 - 2\beta\eta\phi + 2\gamma\eta^5\psi + 4\beta\psi\eta^2 - 4\gamma\phi\eta^4] + 8(\gamma\eta^3 + \beta)\eta^2 x - 8(\beta + \gamma\eta^3)\eta y = 0, \tag{15}$$

which envelopes the cone of Segre.

NATIONAL UNIVERSITY OF CHEKIANG

ON A REPRESENTATION IN SPACE OF GROUPS OF CIRCLE AND TURBINE TRANSFORMATIONS IN THE PLANE

J. M. FELD

1. **Introduction.** In a previous paper [6][1] the author showed that the oriented lineal elements in the euclidean plane can be mapped continuously and (1, 1) upon the points of *quasi-elliptic*[2] three-space Q_3 so that the whirl-similitude group of turbine[3] transformations in the euclidean plane is represented isomorphically upon the group of projective automorphisms of Q_3. By means of this representation *proper* turbines in the plane are mapped upon those real lines in Q_3 which do not intersect a line L, the real part of the quasi-elliptic absolute. It is the purpose of this note to investigate this representation analytically and to extend it so as to yield a (1, 1) continuous mapping of the turbines (proper and improper) in the *Moebius* plane upon all the lines in projective S_3. By such means we establish the isomorphism between certain groups of projective transformations in space on the one hand, and on the other of Kasner's 15-parameter group of turbine transformations [7] and some of its important subgroups, namely the Moebius, Laguerre, and Lie groups of circle transformations.

Other representations of turbines in space are due to Kasner and DeCicco [7, 8] and to A. Narasinga Rao [9]. The former use a

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[1] The numbers in brackets refer to the bibliography at the end of this paper.

[2] The term *quasi-elliptic space* is due to Blaschke [1, 2]. The absolute of this space is composed of a pair of conjugate imaginary planes $x_3^2+x_4^2=0$ and a pair of conjugate imaginary points $(1:\pm i:0:0)$.

[3] The geometry of turbines was initiated by Kasner [7]. An extensive bibliography on the subject is to be found in [5].

representation of lineal elements upon the points of S_3, similar to that used here; the latter maps turbines in the plane upon the points of S_4, and shows thereby that Kasner's group and its subgroups are isomorphic with certain groups of projectivities in S_4. Kasner and DeCicco do not indicate how their representation would map the groups of Moebius, Laguerre, and Lie in S_3.

2. **The mapping $\mathfrak{M}$.** We shall need the following definitions.

DEFINITIONS. A *proper turbine* is a series of ∞^1 oriented lineal elements the points of which lie on a cycle (oriented circle) and the directions of which are inclined at the same angle to the direction of the cycle.

An *improper turbine* is a series of ∞^1 oriented parallel lineal elements the points of which lie on a straight line.

A *proper flat field* is a set of ∞^2 lineal elements which lie on the ∞^1 cycles having a given lineal element (the center of the flat field) in common.

An *improper flat field* is a set containing all of the ∞^2 lineal elements parallel to a given direction.

The proper turbines include the point turbines (=point cycles); the improper turbines include the spears (=oriented lines).

Let the oriented lineal element $\mathfrak{e}$ passing through the point x, y in the xy-plane have the direction θ. Let $z=x+iy$, $\bar{z}=x-iy$, $\zeta=e^{i\theta}$, $0\leqq\theta<2\pi$. Let the symbol (z, ζ) represents $\mathfrak{e}$; z and ζ shall be called the coordinates of $\mathfrak{e}$. In terms of these coordinates a proper turbine has an equation [6] of the form

$$z - l = r\zeta. \tag{2.1}$$

The point l is the center of the turbine and $|r|$ is the length of its radius. We shall represent the turbine (2.1) by the symbol $[l, r]$; l, r shall be called its coordinates. Turbines $[l, r]$ such that $r+\bar{r}=0$ are cycles, and those such that $r=0$ are point turbines.

A proper flat field having the lineal element (s, σ) for its center is given by the equation [6]

$$z - s = (\bar{z} - \bar{s})\sigma\zeta. \tag{2.2}$$

Let the symbol $\{s, \sigma\}$ represent this flat field. Two proper flat fields $\{s, \sigma\}$ and $\{s', \sigma'\}$ such that $\sigma\neq\sigma'$ determine a proper turbine $[l, r]$ where

$$l = (s\sigma' - s'\sigma)/(\sigma' - \sigma), \qquad r = (s - s')/(\bar{\sigma}' - \bar{\sigma}). \tag{2.3}$$

Let a point $\mathfrak{p}$ in S_3 have the homogeneous coordinates $x_1:x_2:x_3:x_4$.

The representation $\mathfrak{M}$ mapping $(z, \zeta) \rightleftarrows p$ continuously and (1, 1) is given by the equations

$$(2.4) \qquad z = 2(x_1 + ix_2)/(x_4 + ix_3), \qquad \zeta = (x_4 - ix_3)/(x_4 + ix_3)$$

and

$$(2.4^*) \qquad \begin{aligned} \rho x_1 &= i(z\bar{\zeta} + \bar{z}), & \rho x_2 &= z\bar{\zeta} - \bar{z}, \\ \rho x_3 &= 2(\bar{\zeta} - 1), & \rho x_4 &= 2i(1 + \bar{\zeta}). \end{aligned}$$

These equations define $\mathfrak{M}$ for all lineal elements in the euclidean plane and all points in S_3 except those on the line $L: x_3 = x_4 = 0$. The points of any plane $u_i x_i = 0$ such that $u_2 + iu_1 \neq 0$, that is, a plane that does not pass through L, are mapped upon the lineal elements of a proper flat field $\{s, \sigma\}$ where

$$(2.5) \qquad s = -2(u_3 + iu_4)/(u_2 + iu_1), \qquad \sigma = (u_2 - iu_1)/(u_2 + iu_1),$$

$$(2.5^*) \qquad \begin{aligned} \rho u_1 &= 2i(1 - \bar{\sigma}), & \rho u_2 &= 2(\bar{\sigma} + 1), \\ \rho u_3 &= -(s\bar{\sigma} + \bar{s}), & \rho u_4 &= i(s\bar{\sigma} - \bar{s}). \end{aligned}$$

Any plane u that does pass through L is mapped upon an improper flat field composed of ∞^2 parallel lineal elements (z, ζ) where

$$(2.6) \qquad \zeta = (u_3 + iu_4)/(u_3 - iu_4).$$

Let $\mathfrak{g}$ be any line in S_3 except L. Let the Plücker axial coordinates of $\mathfrak{g}$ be $q_{ij} = u_i v_j - u_j v_i$. Let us assume, as we may, that neither of the planes u and v passes through L. Evidently $\mathfrak{g}$ is mapped by $\mathfrak{M}$ upon a series of ∞^1 lineal elements common to the proper flat fields $\{s, \sigma\}$ and $\{s', \sigma'\}$ which correspond respectively to u and v. Such a series is a proper turbine when $\sigma \neq \sigma'$ and an improper turbine when $\sigma = \sigma'$.[4] Consequently, the proper turbines in the euclidean plane correspond (1, 1) to those real lines in S_3 which do not intersect L, and the improper turbines correspond (1, 1) to the real lines in S_3 that do intersect L; line L itself so far has no $\mathfrak{M}$-image. The correspondence between turbines and lines is given by the equations:

$$(2.7) \qquad \begin{aligned} \rho q_{12} &= 4i(\bar{\sigma}' - \bar{\sigma}), \\ \rho q_{13} &= -i[(s - s')\bar{\sigma}\bar{\sigma}' - (\bar{s} - \bar{s}') + (\bar{s} + s')\bar{\sigma}' - (s + \bar{s}')\bar{\sigma}], \\ \rho q_{14} &= -(s - s')\bar{\sigma}\bar{\sigma}' - (\bar{s} - \bar{s}') + (s - \bar{s}')\bar{\sigma} - (s' - \bar{s})\bar{\sigma}', \\ \rho q_{34} &= -i(\bar{s}s'\bar{\sigma}' - s\bar{s}'\bar{\sigma}), \\ \rho q_{42} &= i[(s - s')\bar{\sigma}\bar{\sigma}' - (\bar{s} - \bar{s}') + (s + \bar{s}')\bar{\sigma} - (\bar{s} + s')\bar{\sigma}'], \\ \rho q_{23} &= (s - s')\bar{\sigma}\bar{\sigma}' + (\bar{s} - \bar{s}') + (s - \bar{s}')\bar{\sigma} + (\bar{s} - s')\bar{\sigma}'. \end{aligned}$$

[4] $\sigma = \sigma'$ if and only if $\mathfrak{g}$ intersects L.

With the aid of equations (2.3) we find that the proper turbine determined by the flat fields $\{s, \sigma\}$ and $\{s', \sigma'\}$ such that $\sigma \neq \sigma'$ is a cycle when $(s-s')\sigma\sigma' = s-s'$, and a point turbine when $s=s'$. If $\sigma=\sigma'$ the pair of flat fields determines an improper turbine, which, in the event that $(s-s')\sigma^2 = s-s'$, becomes a spear. From (2.7) we find that to the cycles and spears in the euclidean plane correspond (1, 1) all but one of the lines of the linear complex

$$\mathcal{A}: \qquad q_{13} - q_{42} = 0;$$

the excepted line is L $(q_{12}:q_{13}:q_{14}:q_{34}:q_{42}:q_{23}=0:0:0:1:0:0)$. To the point turbines in the euclidean plane correspond (1, 1) all the lines but one, namely L, of the rotationally symmetric elliptic congruence

$$\mathcal{B}: \qquad q_{13} - q_{42} = 0, \qquad q_{14} - q_{23} = 0,$$

which has the lines $x_1+ix_2=x_3-ix_4=0$ and $x_1-ix_2=x_3+ix_4=0$ as directrices.

To make $\mathfrak{M}$ (1, 1) for all the lines of S_3, we therefore close the euclidean plane by adjoining one ideal point turbine at infinity t_∞, which shall serve as the $\mathfrak{M}$-image of L. To the point turbines of the closed (Moebius) plane there now correspond continuously and (1, 1) the lines of $\mathcal{B}$, and to the cycles and spears of the Moebius plane there correspond (1, 1) the lines of $\mathcal{A}$.

To the improper turbines correspond (1, 1) the lines of the special complex $q_{12}=0$; this complex has the line L for its directrix. To the spears, therefore, correspond (1, 1) the lines of the special congruence

$$\mathcal{C}: \qquad q_{13} - q_{42} = 0, \qquad q_{12} = 0.$$

Let the terms *Moebius cycle*, *Laguerre cycle*, and *Lie cycle* designate respectively (1) a cycle with nonzero radius or a spear, (2) any cycle except t_∞, (3) any cycle or spear in the Moebius plane. Since the Moebius group of circle transformations in the plane, regarded as an extended group of contact transformations, transforms point turbines into point turbines and Moebius cycles into Moebius cycles, its $\mathfrak{M}$-image in S_3 is a group of ∞^6 collineations that transform the congruence $\mathcal{B}$ into itself and also transform the linear complex $\mathcal{A}$ into itself. The large (seven-parameter) Laguerre group in the plane is a group of contact transformations that convert spears into spears and Laguerre cycles into Laguerre cycles; consequently, this Laguerre group is isomorphic with a group of ∞^7 collineations in S_3 that leave invariant both the special congruence $\mathcal{C}$ and the complex $\mathcal{A}$. Lie's 10-parameter group of contact transformations transforming Lie cycles into Lie cycles is isomorphic with the group of ∞^{10} collineations in S_3

that leave $\mathcal{A}$ invariant. Kasner's 15-parameter group of turbine transformations in the plane is isomorphic with the group of projectivities (collineations and correlations) in S_3.

Various other representations of the Moebius, Laguerre, and Lie groups have been given, for instance in [3, 4].

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ON THE PATHS WITH MONGE'S EQUATIONS OF THE SECOND DEGREE AS CONDITIONS OF INTERSECTION

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1. Introduction. The problem on the conditions of intersection of a family of curves was suggested by S. Lie[2] and systematically studied by F. Engel.[3] Under the latter's initiative extensive researches have been made on the subject.[4] It is the aim of the present paper to point out the relations between this type of problems and the geometry of paths, thus obtaining the geometrical interpretation of some important invariants of a system of paths. To be definite, we consider in an $(n+1)$-dimensional space a $2n$-parameter family of paths. By using Pfaffian forms, we can define the tensor of the lowest order of the paths in a simple way. This tensor is essentially the one which plays a rôle in the so-called inverse problem of the calculus of variations.[5] It turns out that its vanishing is a necessary and sufficient condition for the conditions of intersection of neighboring paths to be given by a system of Monge's equations of the second degree.

2. The tensor. Let (x, y^i) $(i=1, \cdots, n)$ be coordinates of an $(n+1)$-dimensional space. A $2n$-parameter family of paths is defined by a system of ordinary differential equations of the form

$$(1)\quad y^{ij} = F^i(x, y^j, y^{ij}), \quad i, j = 1, \cdots, n, \quad y^{ij} = \frac{dy^i}{dx}, \quad y^{ij} = \frac{d^2y^i}{dx^2},$$

which can also be written in a Pfaffian system

$$(2)\qquad dy^i - p^i dx = 0, \qquad dp^i - F^i(x, y^j, p^j)dx = 0,$$

where p^i are auxiliary variables. To simplify our calculations we introduce the following Pfaffian forms

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[1] The author wishes to express his thanks to Professor Shiing-shen Chern for suggesting this problem.

[2] S. Lie and G. Scheffers, *Geometrie der Berührungstransformationen*, vol. I, Leipzig, 1896, p. 269.

[3] F. Engel, *Eine neue Methode zur Invariantentheorie der Differentialgleichungen*, Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften vol. 57 (1905) pp. 161–232.

[4] P. Steinbach, *Pfaffsche Gleichungen als Schnittbedingung von Kurvenscharen*, Mitteilungen des Mathematischen Seminars der Universität Giessen, vol. 2, no. 18, 1930.

[5] J. Douglas, *Solution of the inverse problem of the calculus of variations*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) pp. 631–637.

$$\begin{aligned}\omega^i &= dy^i - p^i dx,\\ \pi^i &= dp^i - F^i dx - (\partial F^i/\partial p^k + \delta^i_k H)(dy^k - p^k dx)/2,\end{aligned} \tag{3}$$

the function $H=H(x, y^i, p^i)$ being any particular solution of the partial differential equation

$$2n\frac{dH}{dx} + nH^2 - 4\frac{\partial F^i}{\partial y^i} - \frac{\partial F^i}{\partial p^j}\frac{\partial F^j}{\partial p^i} + 2\frac{d}{dx}\left(\frac{\partial F^i}{\partial p^i}\right) = 0. \tag{4}$$

Here, as throughout this paper, Latin indices run from 1 to n, repeated indices denote summation, and the operator d/dx denotes total differentiation with respect to x along any of the paths, that is,

$$d/dx = \partial/\partial x + p^i\partial/\partial y^i + F^i\partial/\partial p^i.$$

By a simple calculation we find[6]

$$\begin{aligned}(\omega^i)' &= [dx\pi^i] + \frac{1}{2}\left(\frac{\partial F^i}{\partial p^k} + \delta^i_k H\right)[dx\omega^k],\\ (\pi^i)' &= Q^i_j[dx\omega^j] + \frac{1}{2}\left(\frac{\partial F^i}{\partial p^j} - \delta^i_j H\right)[dx\pi^j]\\ &\quad + \frac{1}{2}\left(\frac{\partial^2 F^i}{\partial p^k\partial p^l} + \delta^i_k\frac{\partial H}{\partial p^l}\right)[\omega^k\pi^l] + Q^i_{jk}[\omega^j\omega^k],\end{aligned} \tag{5}$$

where

$$\begin{aligned}Q^i_j &= \frac{\partial F^i}{\partial y^j} + \frac{1}{4}\frac{\partial F^i}{\partial p^k}\frac{\partial F^k}{\partial p^j} - \frac{1}{2}\frac{d}{dx}\left(\frac{\partial F^i}{\partial p^j}\right)\\ &\quad - \frac{1}{n}\delta^i_j\left\{\frac{\partial F^m}{\partial y^m} + \frac{1}{4}\frac{\partial F^m}{\partial p^l}\frac{\partial F^l}{\partial p^m} - \frac{1}{2}\frac{d}{dx}\frac{\partial F^m}{\partial p^m}\right\},\end{aligned} \tag{6}$$

while Q^i_{jk} are functions formed from F^i and H whose expressions do not interest us. The n^2 quantities Q^i_j constitute a tensor in the following sense. Let

$$\bar{x} = \bar{x}(x, y^i), \qquad \bar{y}^i = \bar{y}^i(x, y^i)$$

be a point transformation which carries our paths to a system of paths of the space $(\bar{x}, \bar{y}^i)$. Suppose the new paths be defined by the Pfaffian system

$$d\bar{y}^i - \bar{p}^i d\bar{x} = 0, \qquad d\bar{p}^i - \bar{F}^i(\bar{x}, \bar{y}^i, \bar{p}^i)d\bar{x} = 0, \tag{7}$$

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[6] E. Goursat, *Leçons sur le problème de Pfaff*, Paris, 1922, pp. 15–20.

from which the forms $\bar{\omega}^i$, $\bar{\pi}^i$ and the functions $\overline{H}$, $\overline{Q}^i_j$ can be defined in a similar way. Under this assumption there exist functions u, u_i, u^i_j, r^i_j, s^i_j such that

$$(8)\qquad \begin{aligned} d\bar{x} &= u dx + u_i \omega^i, \\ \bar{\omega}^i &= u^i_j \omega^j, \\ \bar{\pi}^i &= r^i_j \pi^j + s^i_j \omega^j. \end{aligned}$$

Taking the bilinear covariants of the above equations and comparing the coefficients on both sides, we obtain

$$(9)\qquad \begin{aligned} r^i_j &= \frac{1}{u} s^i_j, \\ \frac{du^i_j}{dx} &= u s^i_j + \frac{1}{2} u u^k_j \left(\frac{\partial \overline{F}^i}{\partial \bar{p}^k} + \delta^i_k \overline{H}\right) - \frac{1}{2} u^i_l \left(\frac{\partial F^l}{\partial p^j} + \delta^l_j H\right), \\ \frac{d}{dx}\left(\frac{1}{u} u^i_j\right) &= \frac{1}{2} u^l_j \left(\frac{\partial \overline{F}^i}{\partial \bar{p}^l} - \delta^i_l \overline{H}\right) - \frac{1}{2u} u^i_l \left(\frac{\partial F^l}{\partial p^j} - \delta^l_j H\right) + s^i_j, \\ \frac{d}{dx} s^i_k &= u u^j_k \overline{Q}^i_j - \frac{1}{u} u^i_j Q^j_k + \frac{u}{2} s^j_k \left(\frac{\partial \overline{F}^i}{\partial \bar{p}^j} - \delta^i_j \overline{H}\right) \\ &\quad - \frac{1}{2} s^i_j \left(\frac{\partial F^j}{\partial p^k} + \delta^j_k H\right). \end{aligned}$$

Let v^i_j and s be defined by the equations

$$u^i_j v^j_k = \delta^i_k, \qquad s = s^i_j v^j_i,$$

δ^i_k being the well known Kronecker symbol. Then the first three sets of equations of (9) give

$$(10)\qquad s^j_i = \frac{s}{n} u^j_i,$$

while the last set gives, after the elimination of ds^j_i/dx,

$$(11)\qquad u^i_j \left(\frac{ds}{dx} + \frac{s^2 u}{n} + su\overline{H}\right) = u u^k_j \overline{Q}^i_k - \frac{1}{u} u^i_k Q^k_j.$$

Multiplying (11) by v^j_i and contracting, we get

$$ds/dx + s^2 u/n + su\overline{H} = 0,$$

since $\overline{Q}^i_i = Q^i_i = 0$. Finally, we have $\overline{Q}^i_j = (1/u^2) u^i_k Q^k_l v^l_j$, which shows

that the quantities Q_j^i undergo a linear transformation. The vanishing of Q_j^i has therefore a geometrical meaning.

The quantities Q_j^i differ slightly from similar ones employed by Douglas.[5] The condition $Q_i^i=0$ makes them obey a simpler transformation law.

3. **Conditions of intersection of paths.** Let the system of finite equations defining the family of paths be

$$y^i = G^i(x, C^\alpha), \qquad \alpha = 1, \cdots, 2n. \tag{12}$$

We can consider C^α as coordinates in a $2n$-dimensional space R_{2n} whose points are our paths. Paths are said to be neighboring if the corresponding points are infinitely near. Let Γ and Γ' be two neighboring paths defined by the parameters C^α and $C^\alpha+\delta C^\alpha$ respectively. In order that they intersect, we must have

$$\delta y^i = (\partial G^i/\partial C^\alpha)\delta C^\alpha = 0, \qquad \alpha = 1, \cdots, 2n. \tag{13}$$

The system of $n-1$ equations obtained by eliminating x from (13) gives the conditions of intersection. The $n-1$ equations are independent, otherwise the paths intersecting a given path would depend on more than $n+1$ essential parameters, which case is excluded.

We proceed to show how the conditions of intersection can be treated most simply by our symbolism. Let us introduce two differential operations d and δ, the operation d denoting the variation along the path, while δ denotes the change from one path to the other. By definition they satisfy the conditions

$$dx \neq 0, \qquad dC^\alpha = 0, \qquad \delta x = 0, \qquad \delta C^\alpha \neq 0,$$

so that

$$\begin{aligned}
&\omega^i(d) = 0, \qquad \pi^i(d) = 0, \\
&\omega^i(\delta) = \delta y^i = (\partial G^i/\partial C^\alpha)\delta C^\alpha, \\
&\pi^i(\delta) = \delta p^i - \frac{1}{2}\left(\frac{\partial F^i}{\partial p^j} + \delta_j^i H\right)\delta y^j \\
&\qquad = \left\{\frac{\partial}{\partial C^\alpha}\left(\frac{\partial G^i}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial F^i}{\partial p^j} + \delta_j^i H\right)\frac{\partial G^j}{\partial C^\alpha}\right\}\delta C^\alpha, \\
&\qquad\qquad \alpha = 1, \cdots, 2n.
\end{aligned} \tag{14}$$

According to the theory of completely integrable Pfaffian systems, the last $2n$ equations can be solved in terms of δC^α, thus expressing δC^α as linear combinations of $\omega^i(\delta)$ and $\pi^i(\delta)$.

Suppose that the system giving the conditions of intersection consists only of Monge's equations of the second degree:

$$(15)\qquad A^{\epsilon}_{\alpha\beta}(C^{\gamma})\delta C^{\alpha}\delta C^{\beta} = 0, \qquad \alpha, \beta, \gamma = 1, \cdots, 2n; \epsilon = 1, \cdots, n-1.$$

We express the left-hand side in terms of $\omega^i(\delta)$ and $\pi^i(\delta)$, getting

$$(16)\qquad A^{\epsilon}_{\alpha\beta}(C^{\gamma})\delta C^{\alpha}\delta C^{\beta} = a^{\epsilon}_{ij}\omega^{i}(\delta)\omega^{j}(\delta) + b^{\epsilon}_{ij}\omega^{i}(\delta)\pi^{j}(\delta) + e^{\epsilon}_{ij}\pi^{i}(\delta)\pi^{j}(\delta),$$
$$\alpha, \beta, \gamma = 1, \cdots, 2n, \ \epsilon = 1, \cdots, n-1,$$

where the a's, b's, and e's are functions of x and C^{α} and, for unicity of the coefficients, we can assume

$$(17)\qquad a^{\epsilon}_{ij} = a^{\epsilon}_{ji}, \qquad e^{\epsilon}_{ij} = e^{\epsilon}_{ji}.$$

Since (15) is evidently satisfied when $\omega^i(\delta)=0$, e^{ϵ}_{ij} must vanish, so that (16) becomes

$$(16')\qquad A^{\epsilon}_{\alpha\beta}(C^{\gamma})\delta C^{\alpha}\delta C^{\beta} = a^{\epsilon}_{ij}\omega^{i}(\delta)\omega^{j}(\delta) + b^{\epsilon}_{ij}\omega^{i}(\delta)\pi^{j}(\delta).$$

The left-hand side of the preceding equation is independent of x, so that the same is true of the right-hand side. Thus we have

$$(18)\qquad d\{a^{\epsilon}_{ij}\omega^{i}(\delta)\omega^{j}(\delta) + b^{\epsilon}_{ij}\omega^{i}(\delta)\pi^{j}(\delta)\} = 0.$$

To expand (18) we notice that from (5) we have

$$(19)\qquad \begin{aligned} d\omega^{i}(\delta) &= dx\pi^{i}(\delta) + (\partial F^{i}/\partial p^{k} + \delta^{i}_{k}H)dx\omega^{k}(\delta)/2, \\ d\pi^{i}(\delta) &= Q^{i}_{j}dx\omega^{j}(\delta) + (\partial F^{i}/\partial p^{j} - \delta^{i}_{j}H)dx\pi^{j}(\delta)/2. \end{aligned}$$

Substituting these expressions into (18) we get

$$\left\{da^{\epsilon}_{ik} + \left[a^{\epsilon}_{ij}\left(\frac{\partial F^{j}}{\partial p^{k}} + \delta^{j}_{k}H\right) + b^{\epsilon}_{ij}Q^{j}_{k}\right]dx\right\}\omega^{i}(\delta)\omega^{k}(\delta) + b^{\epsilon}_{ij}dx\pi^{i}(\delta)\pi^{j}(\delta)$$
$$+ \left\{db^{\epsilon}_{ik} + \frac{1}{2}\left(4a^{\epsilon}_{ik} + b^{\epsilon}_{ij}\frac{\partial F^{j}}{\partial p^{k}} + b^{\epsilon}_{jk}\frac{\partial F^{j}}{\partial p^{i}}\right)dx\right\}\omega^{i}(\delta)\pi^{k}(\delta) = 0,$$
$$\epsilon = 1, 2, \cdots, n-1.$$

It follows that

$$b^{\epsilon}_{ij} + b^{\epsilon}_{ji} = 0,$$

$$db^{\epsilon}_{ik} + \frac{1}{2}\left(4a^{\epsilon}_{ik} + b^{\epsilon}_{ij}\frac{\partial F^{j}}{\partial p^{k}} + b^{\epsilon}_{jk}\frac{\partial F^{j}}{\partial p^{i}}\right)dx = 0,$$

$$(20)\qquad 2da^{\epsilon}_{ik} + \left\{a^{\epsilon}_{ij}\left(\frac{\partial F^{j}}{\partial p^{k}} + \delta^{j}_{k}H\right)\right.$$
$$\left. + a^{\epsilon}_{kj}\left(\frac{\partial F^{j}}{\partial p^{i}} + \delta^{j}_{i}H\right) + b^{\epsilon}_{ij}Q^{j}_{k} + b^{\epsilon}_{kj}Q^{j}_{i}\right\}dx = 0.$$

On account of the symmetry of a^ϵ_{ik} and the skew-symmetry of b^ϵ_{ik} in the lower indices the second set of equations of (20) gives

$$a^\epsilon_{ik} = 0. \tag{21}$$

The last two equations of (20) then become

$$db^\epsilon_{ik} + \frac{1}{2}\left(b^\epsilon_{ij}\frac{\partial F^j}{\partial p^k} + b^\epsilon_{jk}\frac{\partial F^j}{\partial p^i}\right)dx = 0, \tag{22}$$

$$b^\epsilon_{ij}Q^j_k + b^\epsilon_{kj}Q^j_i = 0.$$

As a conclusion of our discussions we see that the existence of b^ϵ_{ij} satisfying (22) is a necessary condition for the family of paths to have the prescribed property, that is, the conditions of intersection be given by a system of Monge's equations of the second degree.

4. **The main theorem.** Our main theorem can be stated as follows:

THEOREM. *The conditions of intersection of neighboring paths of the family defined by* (2) *consist only of Monge's equations of the second degree if and only if all the components of the tensor Q^i_j vanish.*

In order to prove the theorem we first establish two simple lemmas.

LEMMA 1. *Let K^ϵ be $n-1$ skew-symmetric matrices of order n whose first rows are linearly independent, and R be any matrix of the same order. Then the equations*

$$K^\epsilon R - R'K^\epsilon = 0, \qquad \epsilon = 1, \cdots, n-1, \tag{23}$$

R' = transpose of R,

imply that R is a scalar matrix.

PROOF. By hypothesis, there exist $n-1$ linear combinations C^ϵ of K^ϵ having their first rows identical to

$$(0, 1, 0, \cdots, 0, 0), (0, 0, 1, \cdots, 0, 0), \cdots, (0, 0, \cdots, 0, 1)$$

respectively. As the K's satisfy (23), so do all the C's, that is,

$$C^\epsilon R - R'C^\epsilon = 0.$$

By actually writing out the elements of the matrices, we can justify this lemma by a short calculation.

LEMMA 2. *Let $b^\epsilon_{ij}x^iy^j = 0$, $\epsilon = 1, \cdots, n-1$, be $n-1$ independent equations in x^i, y^i, with skew-symmetric coefficients*

$$b^\epsilon_{ij} + b^\epsilon_{ji} = 0.$$

Let Q be a square matrix satisfying

$$B^{\epsilon}Q - Q'B^{\epsilon} = 0, \tag{24}$$

where

$$B^{\epsilon} = (b^{\epsilon}_{ij}).$$

Then Q is a scalar matrix.

PROOF. We can find x^i_0 such that the $n-1$ linear equations in y^i,

$$b^{\epsilon}_{ij}x^i_0y^j = 0,$$

are linearly independent. Let P be any nonsingular matrix with x^i_0 as its first row. The first rows of PB^{ϵ} are

$$(b^{\epsilon}_{i1}x^i_0,\ b^{\epsilon}_{i2}x^i_0,\ \cdots,\ b^{\epsilon}_{in}x^i_0),$$

respectively, and hence are linearly independent. It is easy to see that the matrices $PB^{\epsilon}P'$ satisfy all the hypotheses for K^{ϵ} in Lemma 1. By writing (24) in the form

$$(PB^{\epsilon}P')(P'^{-1}QP') - (P'^{-1}QP')'(PB^{\epsilon}P') = 0$$

it follows that $P'^{-1}QP'$, and hence Q, is a scalar matrix.

PROOF OF THE MAIN THEOREM. Suppose that the conditions of intersection of neighboring paths of the family consist only of Monge's equations of the second degree. There exist b^{ϵ}_{ij} $(\epsilon=1, \cdots, n-1)$ satisfying (22) and having the property that the $n-1$ equations

$$b^{\epsilon}_{ij}\omega^i(\delta)\pi^j(\delta) = 0$$

are independent. From Lemma 2 it follows that

$$Q^i_j = (1/n)\delta^i_jQ^l_l = 0.$$

Conversely, suppose $Q^i_j=0$. We regard $b^{\epsilon}_{ik} = -b^{\epsilon}_{ki}$ as functions of x, C^{α} and consider the system of $(n-1)n/2$ partial differential equations

$$\partial b_{ik}/\partial x + (1/2)(b_{ij}\partial F^j/\partial p^k + b_{jk}\partial F^j/\partial p^i) = 0,$$

where

$$\frac{\partial F^i}{\partial p^i} = \frac{\partial F^i(x, G^m, \partial G^m/\partial x)}{\partial p^i}.$$

Let b^{ϵ}_{ij} be $n-1$ sets of integrals satisfying the initial conditions that all $(b_{ij})_{x=x_0}$ vanish except

$$(b^{\epsilon}_{1\,\epsilon+1})_{x=x_0} = -(b^{\epsilon}_{\epsilon+1\,1})_{x=x_0} = 1.$$

The quantities b_{ij}^{*} so obtained not only satisfy (22) but also possess the property that the equations

$$b_{i,j}^{*}\omega^{i}(\delta)\pi^{j}(\delta) = 0$$

are independent. That these equations give the conditions of intersection of neighboring paths follows from our discussion. Therefore the theorem is proved.

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ERRATA, VOLUME 50

Page 31, line 7. Instead of "Abstract 50-1-14" read "Abstract 50-1-13."

R. v. Mises, *Integral theorems in three-dimensional potential flow.* Page 603, display (17) should read $\int[r(q\cdot n)-(r/2)\times(z\times n)dS]$ $=\int[r(q\cdot n)-q(r\cdot n)/2+n(r\cdot q)/2]dS=0$.

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